Lecture 5: Auctions
Auctions – A Brief History

- **auctions**: an alternative to take-it-or-leave-it pricing, competition of potential buyers

  *Babylonian empire (500 BC)*: auctions of women for marriage

  *Roman empire*: auctions to liquidate the assets of debtors whose property had been confiscated

  *17th – 18th century, Europe*: auctions to sell pieces of art, the birth of many auction houses that still work today:
  - 1674: Stockholm Auction House
  - 1744: Sotheby's
  - 1766: Christie's
  - ...

Today: online auctions for all kinds of things (*eBay, eBid, Aukro, …*)
Types of Auctions

Famous types of auctions:

- **English auction** (a.k.a. *open ascending price auction*)
  - the most widespread auction type (the typical art auctions)
  - open bidding, bidders know the others' bids
  - various rule modifications (ending rules – e.g. “auction by candle”)

- **Dutch auction** (a.k.a. *open descending price auction*)
  - the auctioneer cries out gradually descending price bids, the first one to accept the price is the buyer
  - cut flower sales in the Netherlands, perishable goods (fish, tobacco)

- **Envelope auction** (a.k.a. *first-price sealed-bid auctions*)
  - bidders can only submit one bid each (typically, in a sealed envelope)
  - the sale of real estate and securities (used a lot in the post-communist countries)

- **Vickrey auction** (a.k.a. *second-price sealed-bid auctions*)
  - “designed” by William Vickrey in 1961
  - used to auction off collectible stamps
Basic classification of auction rules:

- **ascending/descending**
  - the direction of bid increments

- **open/sealed-bid**
  - open – bidders submit the bids publicly and after one another
  - sealed-bid – bidders submit the bids secretly and simultaneously

- **first-price/second-price**
  - winner pays the highest/second-highest bid

- **single object/multi-object auction**
  - number of objects auctioned at the same time

- **reserve/no-reserve**
  - the seller can state a reserve price – the minimum price of the auctioned object
  - no-reserve auctions – can attract more bidders (?)
Some other auction types:

- **All-Pay Auctions**
  - used for charity auctions
  - various schemes (paying all bids or paying all increments + the whole of the winning bid)

- **Auctions with Buyout Option**
  - the seller can state a buyout price – for immediate purchase

- **Combinatorial Auctions**
  - multi-object auctions, bidding for bundles of objects
  - ferry lines, airport landing slots (it only makes sense to have bundles)
  - winner determination problem, preference expression problems

- **Online Timeshift Auctions**
  - fixed-time English type
  - aim: make bidders bid before the closing timeshift interval
Basic auction rules:
- bidders submit one bid each
- bids are sealed (= secret) and simultaneous
- first-price auction (winner pays the highest bid)

Additional assumptions (for mathematical modelling):
- *two bidders* only (can be relaxed easily; however, we want to use bi-matrix games as the modelling tool); bidders = *investor 1* and 2
- investors possess information about the subjective value of each of the *n* auctioned objects: $s_1, s_2, \ldots, s_n$
- total amounts the bidders intend to invest are known: $I_1, I_2$
- there's a reserve price for each object: $d_1, d_2, \ldots, d_n$
  (we assume that $s_i \geq d_i$ for $i = 1, 2, \ldots, n$)
- in case of equal non-zero bids, the object in question is sold to each of the investors with a probability of $\frac{1}{2}$ (a fair lottery)
Modelling the auction as a normal-form game:

- *strategy spaces* of the players:

\[ X = \left\{ x = (x_1, x_2, \ldots, x_n); \sum_{i=1}^{n} x_i = I_1, \ x_i \in [d_i, s_i] \cup \{0\} \right\}, \]

\[ Y = \left\{ y = (y_1, y_2, \ldots, y_n); \sum_{i=1}^{n} y_i = I_2, \ y_i \in [d_i, s_i] \cup \{0\} \right\}. \]
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- \( y_i = \) player 2’s bid for object \( i \)
- sum of the bids = the intended amount of total investment
- it makes sense to bid either 0, or anything between the reserve price and the value of the object
in general, these strategy spaces can be *infinite*

however, it is usually required that the bids must be an integer multiple of a specified monetary unit → *finite* strategy spaces (which enables the bimatrix approach)

**Example 1:**

- three objects \((n = 3)\)
- values: \(s_1 = 40, s_2 = 22, s_3 = 20\)
- reserve price 10 for all objects \((d_1 = d_2 = d_3 = 10)\)
- total investment: \(I_1 = 20, I_2 = 10\)
- bids must be integer multiples of 10
- strategy spaces (expressed in multiples of 10 for brevity):

\[
X = \{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\} \quad \leftarrow 6 \text{ pure strategies, sum } = 2
\]

\[
Y = \{(1,0,0),(0,1,0),(0,0,1)\} \quad \leftarrow 3 \text{ pure strategies, sum } = 1
\]
payoff functions:

- in order to formulate the payoff functions for both players, we introduce the following functions:

\[ \alpha(x, y) = \begin{cases} 
1 & \text{for } x > y, \\
\frac{1}{2} & \text{for } x = y, \\
0 & \text{for } x < y \text{ or } x = y = 0.
\end{cases} \]

\[ \beta(x, y) = \begin{cases} 
1 & \text{for } x < y, \\
\frac{1}{2} & \text{for } x = y, \\
0 & \text{for } x > y \text{ or } x = y = 0.
\end{cases} \]

- note: \(\alpha(x_i, y_i)\) is the probability of player 1 obtaining object \(i\) and \(\beta(x_i, y_i)\) is the probability of player 2 obtaining object \(i\).

- if player 1 obtains \(i\)th object, his total profit rises by \(s_i - x_i\).

- payoff functions express the expected total payoff for the players:

\[ Z_1(x, y) = \sum_{i=1}^{n} (s_i - x_i) \alpha(x_i, y_i), \quad Z_2(x, y) = \sum_{i=1}^{n} (s_i - y_i) \beta(x_i, y_i). \]
Example 1 (cont’d):

- assume \( x = (20,0,0) \) and \( y = (10,0,0) \), then:
  - player 1 wins object 1
  - player 2 wins nothing
  \[ Z_1(x,y) = s_1 - x_1 = 40 - 20 = 20 \]
  \[ Z_2(x,y) = 0 \]
- using the formula for \( Z_1 \):
  \[ Z_1(x,y) = \sum (s_i - x_i) \alpha(x_i,y_i) = (40 - 20) \times 1 + (22 - 0) \times 0 + (20 - 0) \times 0 \]
  (legend: values, bids, probabilities)

- in case \( x = (10,10,0) \) and \( y = (10,0,0) \): player 1 wins object 2; object 1 is decided by a toss of a coin:
  \[ Z_1(x,y) = (40 - 10) \times \frac{1}{2} + (22 - 10) \times 1 + 0 = 27 \]
  \[ Z_2(x,y) = (40 - 10) \times \frac{1}{2} + 0 + 0 = 15 \]
Multi-Object Sealed-Bid Auctions (cont’d)

- game-theoretical solution to the bidding problem – NE again:

  A strategy profile \((x^*, y^*)\) with the property that
  
  \[
  Z_1(x, y^*) \leq Z_1(x^*, y^*), \\
  Z_2(x^*, y) \leq Z_2(x^*, y^*)
  \]

  for all \(x \in X\) and \(y \in Y\) is a NE.

- finite strategy spaces → the auction can be modelled as a bimatrix game

- possible outcomes:
  - unique NE in pure strategies
  - multiple NE’s (pure and mixed), no domination
  - multiple NE’s (pure and mixed), one dominates the others
  - no pure NE’s, (mixed NE’s only)
1. Formulate the auction from example 1 as a bimatrix game (i.e., find the payoff matrices for both players, and write them down in a single matrix with double entries).

2. Find the NE of the bimatrix game.

\[
\begin{array}{c|ccc}
\text{Investor 1} & s_1 & s_2 & s_3 \\
\hline
s_1 & 40 & & \\
\hline
s_2 & 22 & & \\
\hline
s_3 & 20 & & \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{Investor 2} & 1,0,0 & 0,1,0 & 0,0,1 \\
\hline
1,2 & 20; 0 & & \\
2,0,0 & 27; 15 & & \\
1,1,0 & & & \\
1,0,1 & & & \\
0,2,0 & & & \\
0,1,1 & & & \\
0,0,2 & & & \\
\end{array}
\]
Exercise 1: Unique Pure-Strategy NE

1. Formulate the auction from example 1 as a bimatrix game (i.e., find the payoff matrices for both players, and write them down in a single matrix with double entries).

2. Find the NE of the bimatrix game.

\[
\begin{array}{c|ccc}
\text{Investor 1} & 1 & 0 & 1 \\
\hline
1 & 20 ; 0 & 20 ; 12 & 20 ; 10 \\
2,0,0 & 27 ; 15 & 36 ; 6 & 42 ; 10 \\
1,1,0 & 25 ; 15 & 40 ; 12 & 35 ; 5 \\
1,0,1 & 2 ; 30 & 2 ; 0 & 2 ; 10 \\
0,2,0 & 22 ; 30 & 16 ; 6 & 17 ; 5 \\
0,1,1 & 0 ; 30 & 0 ; 12 & 0 ; 0 \\
0,0,2 & s_1 = 40 & s_2 = 22 & s_3 = 20 \\
\end{array}
\]
Exercise 2: Multiple NE’s – Solvable Case

- consider similar auction as in example 1, only that the values of the object are: \( s_1 = 36 \), \( s_2 = 24 \), \( s_3 = 20 \)

- the payoff matrices are in the following table; find all NE’s for this auction

<table>
<thead>
<tr>
<th>Investor 1 \ Investor 2</th>
<th>1,0,0</th>
<th>0,1,0</th>
<th>0,0,1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,0,0</td>
<td>16 ; 0</td>
<td>16 ; 14</td>
<td>16 ; 10</td>
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<tr>
<td>1,1,0</td>
<td>27 ; 13</td>
<td>33 ; 7</td>
<td>40 ; 10</td>
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<tr>
<td>1,0,1</td>
<td>23 ; 13</td>
<td>36 ; 14</td>
<td>31 ; 5</td>
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<tr>
<td>0,2,0</td>
<td>4 ; 26</td>
<td>4 ; 0</td>
<td>4 ; 10</td>
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<tr>
<td>0,1,1</td>
<td>24 ; 26</td>
<td>17 ; 7</td>
<td>19 ; 5</td>
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<tr>
<td>0,0,2</td>
<td>0 ; 26</td>
<td>0 ; 14</td>
<td>2 ; 0</td>
</tr>
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</table>
Exercise 3: Multiple NE’s

- consider similar auction as in example 1, only that the values of the object are: $s_1 = 26$, $s_2 = 24$, $s_3 = 22$
- find the payoff matrices and all NE’s for this auction

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<thead>
<tr>
<th>Investor 1</th>
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<tbody>
<tr>
<td>2,0,0</td>
<td>6 ; 0</td>
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<tr>
<td>1,1,0</td>
<td>22 ; 8</td>
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<tr>
<td>1,0,1</td>
<td>20 ; 8</td>
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<tr>
<td>0,2,0</td>
<td>4 ; 16</td>
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<tr>
<td>0,1,1</td>
<td><strong>26 ; 16</strong></td>
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<td>0,0,2</td>
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</table>

$s_1 = 26$, $s_2 = 24$, $s_3 = 22$
Exercise 4: No Pure-Strategy NE

- consider similar auction as in example 1, only that the values of the object are: \( s_1 = 80, s_2 = 24, s_3 = 22 \)
- check that there are no pure-strategy NE's for this auction

\[
\begin{array}{c|ccc}
\text{Investor 1} & 1,0,0 & 0,1,0 & 0,0,1 \\
\hline
1,0,0 & 60;0 & 60;14 & 60;12 \\
1,1,0 & 49;35 & 77;7 & 84;12 \\
1,0,1 & 47;35 & 82;14 & 76;6 \\
0,2,0 & 4;70 & 4;0 & 4;12 \\
0,1,1 & 26;70 & 19;7 & 20;6 \\
0,0,2 & 2;70 & 2;14 & 2;0 \\
\end{array}
\]

\( s_1 = 80 \quad s_2 = 24 \quad s_3 = 22 \)
Mixed Strategies in Auctions

- from the Nash Existence Theorem, we know that for every bimatrix game there exists at least one NE in mixed strategies
- finding mixed strategies: procedure based on the Equivalence Theorem:

**Equivalence Theorem.** Let $A$ and $B$ be $m \times n$ matrices with positive elements. The vectors $p^*$ and $q^*$ are non-zero solution of the nonlinear programming problem

\[
\begin{align*}
\text{maximize} & \quad M(p,q) = p^\top (A + B)q - 1_m^\top p - 1_n^\top q \\
\text{subject to} & \quad Aq \leq 1_m, \\
& \quad B^\top p \leq 1_n, \\
& \quad p \geq 0, \\
& \quad q \geq 0.
\end{align*}
\]

if and only if $x^* = bp^*$ and $y^* = aq^*$ represent a mixed-strategy NE of the bimatrix game with matrices $A,B$, where:

\[
\frac{1}{b} = 1_m^\top p^* = \sum p_i, \quad \frac{1}{a} = 1_n^\top q^* = \sum q_i, \quad M(p^*,q^*) = 0.
\]
although we can solve the model using *MS Excel Solver* again, there are several problems:

- non-linear optimization problems may have multiple local extremes, it’s advisable to run the algorithm from *different starting points*
- to solve the auction, we need the equilibrium to be *unique* (or dominant)
- unfortunately, there are no efficient ways of testing the uniqueness of a mixed strategy equilibrium

- a “relatively reliable” procedure of finding a mixed-strategy NE:
  - **Step 1**: solve the optimization problem

    
    \[
    \text{maximize } \mathbf{1}_m^T \mathbf{p} + \mathbf{1}_n^T \mathbf{q} = \sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j \text{ subject to } (2),
    \]

    keep the optimal solution from step 1 as the starting point for step 2

  - **Step 2**: solve the problem: maximize $M(p,q)$ subject to (2); denote optimal values of $\mathbf{p}$ and $\mathbf{q}$ as $\mathbf{p}^*$ and $\mathbf{q}^*$

  - **Step 3**: normalize $\mathbf{p}^*$ and $\mathbf{q}^*$ from step 2 in order to get NE mixed strategies $\mathbf{x}^*$ and $\mathbf{y}^*$ (note: normalize a vector = divide by the sum of its elements)
Collusive Auctions

collusion = secret agreement, conspiracy

aim of auctions: generate the maximum revenue for the seller; works only if the bidders compete
→ collusion is usually not accepted by the auction rules

modelling approach: cooperative bimatrix games with transferable payoffs

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Games and Decisions
Jan Zouhar
Collusive Auctions

- in order to find the core of the game, we first need: $v(1)$, $v(2)$, and $v(1,2)$
- finding guaranteed payoffs: *eliminate strictly dominated strategies first!*
  - $v(1) = 60$
  - $v(2) = 7$

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Collusive Auctions (cont’d)

- maximum total payoff = \( v(1, 2) = 96 \)
  (note: dominated strategies are included here!)

- core of the game:
  \( a_1 + a_2 = 96, \)
  \( a_1 \geq 60, \)
  \( a_2 \geq 7. \)

- superadditive effect: \( v(1, 2) - v(1) - v(2) = 96 - 60 - 7 = 29 \)

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