Lecture 2: Matrix Games
Normal (Strategic) Form Games

Main traits:

- **simultaneous moves**
  - players have to make their strategy choices *simultaneously, without knowing the strategies* that have been chosen by the other player(s)

- **common knowledge of available strategies**
  - while there is no information about what other players will actually choose, we assume that the *strategic choices* available to each player *are known by all players*

- **rationality & interdependence**
  - players must think not only about their own best strategic choice but also the best strategic choice of the other player(s)
Normal (Strategic) Form Games

- mathematical notation of a game’s elements:
  - a finite set of agents: \( \{1,2,...,n\} \)
  - strategy spaces (finite or infinite):
    - after selection: a strategy profile: \( x = (x_1, x_2, ..., x_n) \)
  - payoff functions: \( Z_1(x), Z_2(x),..., Z_n(x) \)
    (or: \( Z_1(x_1, x_2, ..., x_n), Z_2(x_1, x_2, ..., x_n),..., Z_n(x_1, x_2, ..., x_n) \))

- infinite strategy spaces: payoff functions typically expressed as mathematical formulas:
  \[
  Z_1(x_1, x_2) = 100 - (x_1 + 2x_2),
  \]
  \[
  Z_2(x_1, x_2) = 150 - (2x_1 + x_2).
  \]

- example: Cournot oligopoly, strategies = quantities supplied
- **finite strategy spaces**: payoffs specified in tables (or matrices)
  - Prisoner’s dilemma revisited:
    - payoffs in two matrices → a *bimatrix game*

<table>
<thead>
<tr>
<th>Player A</th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stay silent</td>
<td></td>
</tr>
</tbody>
</table>
| Stay silent | \(-1, -1\)  
| Betray | \(-10, 0\)  

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
\textbf{A} & \textbf{B} & \\
\hline
Stay silent & \(-1, -1\) & \(-10, 0\) \\
Stay silent & \hline
Betray & \(-10, 0\) & \(-5, -5\) \\
\hline
\end{tabular}
\end{table}
choose a strategy for player 1 in the following bimatrix game:

\[
\begin{array}{c|cccc}
|   & W & X & Y & Z \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5,2</td>
<td>2,6</td>
<td>1,4</td>
<td>1,4</td>
</tr>
<tr>
<td>B</td>
<td>9,5</td>
<td>1,3</td>
<td>0,2</td>
<td>4,8</td>
</tr>
<tr>
<td>C</td>
<td>7,0</td>
<td>2,2</td>
<td>1,5</td>
<td>5,1</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>3,2</td>
<td>2,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>
\end{array}
\]
Game 1: A Bimatrix Game

- comments on individual strategies:
  - A: no matter what player 2 (opponent) plays, this strategy is worse than or equal to C (i.e., C weakly dominates A)
    → no rational player would ever play A!
  - B: high payoff combinations (B,W) and (B,Z)
    → works only in case players cooperate!
  - B and C: highest sum of possible payoffs (i.e., row sums)
    → best only if the opponent picks her strategy at random!
  - C: looks safe – always gives the highest or second-highest payoff
    → doesn’t take the opponents rationality into account!

- game-theoretic approach: both players rational, aware of the other player’s rationality
  - optimal strategy: D (we’ll be explaining why in the following lectures)
suppose there are just two television networks. Both are battling for shares of viewers (0–100%). Higher shares are preferred (= higher advertising revenues).

■ sum of shares = 100%, i.e. for two players

\[ Z_1(x_1,x_2) + Z_2(x_1,x_2) = \text{const.} \quad \text{for all } (x_1,x_2) \]

■ network 1 has an advantage in sitcoms. If it runs a sitcom, it always gets a higher share than if it runs a game show.

■ network 2 has an advantage in game shows. If it runs a game show it always gets a higher share than if it runs a sitcom.

<table>
<thead>
<tr>
<th></th>
<th>Network 1</th>
<th>Network 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 \ 2</td>
<td>Sitcom</td>
<td>Game show</td>
</tr>
<tr>
<td>Sitcom</td>
<td>55%, 45%</td>
<td>52%, 48%</td>
</tr>
<tr>
<td>Game show</td>
<td>50%, 50%</td>
<td>45%, 55%</td>
</tr>
</tbody>
</table>
Zero-Sum Games

- **zero-sum game**: a special case of constant-sum games

  \[ \text{sum of payoffs} = Z_1 + Z_1 + \ldots + Z_n = 0. \]

- Every constant-sum game has a *strategically equivalent* counterpart in zero-sum games

  - Example: zero-sum version of battle of the networks
  - Payoffs expressed as the difference from the 50/50 share
  - Differences in outcomes unchanged → *strategic equivalence*

  \[
  \begin{array}{c|cc}
  \text{Network 1} & \text{Sitcom} & \text{Game show} \\
  \hline
  \text{Sitcom} & 5\% , -5\% & 2\% , -2\% \\
  \text{Game show} & 0\% , 0\% & -5\% , 5\%
  \end{array}
  \]

  *Note: entries for 1 and 2 always opposite (Z_1 = -Z_2) → no need to write both!*
Matrix Games

- a special case of zero-sum games:
  - a finite set of agents: \{1,2\}
  - strategy spaces (finite): \{X,Y\}
  - strategy profile: \( (x,y) \)
  - payoff functions: \( Z_1(x,y), Z_2(x,y) \)
  - zero-sum payoffs: \( Z_1(x,y) + Z_2(x,y) = 0 \)

- payoffs written in a matrix, typically denoted by \( A \):

\[
A = (a_{ij})_{i=1,...,m}^{j=1,...,n} = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

- \( a_{ij} = \text{the payoff of player 1 for strategy profile } (i,j) \)
  (i.e., player 1 picks \( i \)th strategy and player 2 picks \( j \)th)
Matrix Games

- **example**: battle of the networks (zero-sum version)
  - a matrix game with
    \[
    A = \begin{bmatrix}
    5 & 2 \\
    0 & -5
    \end{bmatrix}
    \]
  - note: in order to know the strategic nature of the game, nothing else needs to be specified (the payoffs and number of strategies of both players are determined by $A$)

<table>
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<th>Network 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1 \ 2</strong></td>
<td>Sitcom</td>
<td>Game show</td>
</tr>
<tr>
<td>Sitcom</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Game show</td>
<td>0</td>
<td>-5</td>
</tr>
</tbody>
</table>
Nash Equilibrium

- one of the most widely used game-theoretical concepts (not only for matrix games)
- best-response approach:
  - determine the “best response” of each player to a particular choice of strategy by the other player (do this for both players)
  - if each player’s strategy choice is a best response to the strategy choice of the other player, we’re in a Nash equilibrium (NE)

“NE is such a combination of strategies that neither of the players can increase their payoff by choosing a different strategy.”

“NE is a solution with the property that whoever of the players chooses some other strategy, he or she will not increase his or her payoff.”
Nash Equilibrium in Matrix Games

- **Mathematical definition:**

  A strategy profile \((x^*, y^*)\) with the property that

  \[ Z_1(x, y^*) \leq Z_1(x^*, y^*) \leq Z_1(x^*, y) \]

  for all \(x \in X\) and \(y \in Y\) is a NE.

Inequality from the definition above explained:

- If player 1 deviates from NE, he/she won’t be any better off

- If player 2 deviates from NE, he/she won’t be any better off
Finding NE’s in Matrix Games

- a matrix game can have 0, 1 or multiple NE’s
- **best-response analysis** (a.k.a. cell-by-cell inspection)
  - network 1’s best response:
    - if network 2 runs a sitcom, network 1’s best response is to run a sitcom. Circle \((S,S)\).
    - if network 2 runs a game show, network 1’s best response is to run a sitcom. Circle \((S,G)\).

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong></td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td><strong>2</strong></td>
<td>0</td>
<td>-5</td>
</tr>
</tbody>
</table>
network 2’s best response:

- if network 1 runs a sitcom, network 2’s best response is to run a game show. Square \((S,G)\).
- if network 1 runs a game show, network 2’s best response is to run a game show. Square \((G,G)\).

- the NE strategy profile is \((S,G)\). (if network 2 plays \(G\), network 1’s best response is \(S\) and vice versa)
from the best-response analysis it follows that a NE is represented by such an element in the payoff matrix that is both...

- ...the *maximum* in its column (player 1’s best response)
- ...the *minimum* in its row (player 2’s best response)

such an element is called a **saddle point** of the matrix

value of a saddle point $= Z_1(x^*,y^*) = \text{value of the game}$

notion of *stability*: neither player has an incentive to deviate from NE
find a NE in the following matrix game:

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>42</td>
<td>10</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>-12</td>
<td>56</td>
<td>2</td>
<td>12</td>
</tr>
</tbody>
</table>
find all NE’s in the following matrix game

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>-2</td>
<td>7</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
Multiple Nash Equilibria

QUIZZ:
Consider a matrix game with payoff matrix $A = (a_{ij})$. Let $a_{27}$ and $a_{43}$ be two NE’s.

a) Is it possible that $a_{27} < a_{43}$?
b) Are $a_{23}$ and $a_{47}$ Nash equilibria as well?

- answers:
  a) no.
  b) yes.

- how to find out: use the basic properties of a saddle point
  *(see next slide)*
**Multiple Nash Equilibria**

*Conclusion:* multiple equilibria always have *equal values* and are placed in “rectangular positions”
Dominated strategies

- **definition:**

  *Strategy* $x_1 \in X$ strictly dominates *strategy* $x_2 \in X$, if

  $Z_1(x_1,y) > Z_1(x_2,y)$ for all $y \in Y$.

  Analogously, $y_1 \in Y$ strictly dominates *strategy* $y_2 \in Y$, if

  $Z_1(x,y_1) < Z_1(x,y_2)$ for all $x \in X$.

  Weak domination is similar, only it admits $Z_1(x_1,y) = Z_1(x_2,y)$ for some $y \in Y$, or $Z_1(x,y_1) = Z_1(x,y_2)$ for some $x \in X$.

- **example:**

  - network 1: $G$ is dominated by $S$
  - network 2: $S$ is dominated by $G
as a *rational* player would *never* play a *dominated strategy*, matrix games can be simplified by deleting the players’ dominated strategies

**Iterative elimination of dominated strategies:**
- elimination of dominated rows → columns → rows → columns → …
- I-know-he-knows-I’m-rational type of thinking

Games and Decisions
example: extended battle of networks ($T = \text{talent show}$)

1. network 2: no dominated strategies
2. network 1: $G$ dominated (by $S$)
3. network 2: $G$ dominated (by $T$)
4. network 1: $T$ dominated (by $S$)
5. network 2: $S$ dominated (by $T$)
- note: sometimes no strategies to eliminate, but still a single NE, as in the example below

### Network 1

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>G</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>5</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>G</td>
<td>6</td>
<td>-5</td>
<td>-4</td>
</tr>
<tr>
<td>T</td>
<td>-2</td>
<td>3</td>
<td>-1</td>
</tr>
</tbody>
</table>

### Network 2

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>G</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>G</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
eliminate dominated strategies in the matrix game from Exercise 2:

<table>
<thead>
<tr>
<th></th>
<th>W</th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>-2</td>
<td>7</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>
in this game, each player has 3 strategies: rock (R), paper (P) and scissors (S).

rules:
- scissors cut paper
- paper wraps rock
- rock crushes scissors
- winner gets €1 from his/her opponent

a) Can this game be modelled as a matrix game?
b) Which strategy would you choose?
c) Are there any saddle points in the matrix?
no saddle point in the payoff matrix, but still there’s a way to play the game

- all strategies “equally good” → the best thing for both players is to choose their strategy at random, with equal probabilities
- even if the other player finds out about the other players’ strategy, he/she can’t use it against him/her
→ switch from *pure strategies* to *mixed strategies*

- pure strategy: the player decides for a certain strategy
- mixed strategy:
  - the player decides about the probabilities of the alternative strategies
  - when the decisive moment comes, he/she makes a random selection of the strategy with the stated probabilities

- even if a matrix game has no NE in pure strategies (i.e., no saddle point of the payoff matrix), it still has a NE in mixed strategies (*always*)

- optimal mixed strategies for RPC game:

\[
\begin{bmatrix}
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3}
\end{bmatrix}, \quad
\begin{bmatrix}
\frac{1}{3} \\
\frac{1}{3} \\
\frac{1}{3}
\end{bmatrix}
\]
LECTURE 2: MATRIX GAMES

Jan Zouhar
Games and Decisions