Lecture 9:
Risk and Uncertainty (cont'd), Utility Theory *p*-intelligent Players

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Games and Decisions

Decisions under Risk

- *risk*: the opponent is a random mechanism that chooses the strategies according to a known probability distribution

  → for each strategy, payoff is a random variable with a known distribution
- *expected value principle*: it's rational to maximize the *expected payoff* (i.e., choose the strategy that yields the maximum expected value of payoffs)
- however, such strategies are often not picked in practice (expected value principle is not *normative*) – see the following exercise

Exercise 1: Three Lotteries

- you were given the opportunity to take part in one of the following lotteries (*A*,*B*, or *C*, see table below); the result all the lotteries is determined by rolling a die

Die roll – result

Lottery

1 \ 2	1	2	3	4	5	6
A	2	6	2	6	2	6
B	-60	0	0	0	0	120
C	3	3	3	3	3	3

1. Which of the lotteries would you choose?
2. What is the *expected payoff* for each of the lotteries?
3. If you wouldn't take part in the lottery with the highest payoff, explain why.
4. Calculate the *variances* of each of the lotteries' outcome.

Exercise 1: Three Lotteries (cont'd)

Variance of a discrete-valued random variable:

Let X be a discrete-valued random variable. *Variance* of X is given by

$$\operatorname{var} X = \sum_x (x - \operatorname{E} X)^2 \cdot \Pr(X = x)$$

	Expectation	Variance
Lottery *A*	4	4
Lottery *B*	10	2,900
Lottery C	3	0

- the reluctance to enter risky lotteries is natural (*risk aversion*)
  - here: risk level expressed in terms of *variance of payoffs*.
- risk aversion can be explained from the strictly rational standpoint using *utility theory*

Exercise 2: St. Petersburg Paradox

- described by Daniel Bernoulli in 1738, a.k.a. *St. Petersburg Lottery*, *Bernoulli's Paradox*
- rules of the lottery:
  - a fair coin is tossed repeatedly, until a *tail* appears, ending the game
  - the pot starts at €1 and is doubled each time a *head* appears; after the game ends, you'll win whatever is in the pot
    - example: T €1
      H-T €2
      H-H-T €4
      $\underbrace{H\text{- … -}H}_{n\text{-times } H}$-$T$ .. €2^n

1. Imagine you have a ticket to play the game (once). For how much would you be willing to sell it (i.e., what price would you ask for it)?
2. What is the expected payoff of the lottery?

Exercise 2: St. Petersburg Paradox (cont'd)

- this is an example of a game where nobody follows the expected payoff principle
- possible outcomes are not limited (in theory, heads can appear any number of times in a row) → average payoff is a weighted sum of infinite number of values:

$$\text{expected payoff} = \underbrace{1\times\frac{1}{2}}^{T} + \underbrace{2\times\left(\frac{1}{2}\right)^2}^{H\text{-}T} + \underbrace{2^2\times\left(\frac{1}{2}\right)^3}^{H\text{-}H\text{-}T} + \ldots + \underbrace{2^n\times\left(\frac{1}{2}\right)^{n+1}}^{n\times H\text{-}T} + \ldots =$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} + \ldots =$$

$$= \infty$$

- Bernoulli's explanation: *utility theory*

Utility Theory

- people do not compare money amounts, but the resulting **utility** (= level of satisfaction)
- monthly wages of both Peter (a teacher) and Paul (a company's CEO) have increased by €500
  - Peter's wage: from €1,000 to €1,500
  - Paul's wage: from €20,000 to €20,500

  → which one of them did the change make happier?
- utility expressed as a *function of monetary amounts*
- units of utility sometimes called **utils**

- Bernoulli's suggestion:
  - if a person's wealth changes, the increments of utility correspond to a *relative change of wealth* (rather than absolute)
    (Peter's wage went up by 50%, while Paul's only by 2,5%)

  → repeated doubling of one's wealth yields constant utility increments

Utility Theory (cont'd)

- the only function with such a property is the *logarithmic function*

$$u(x) = a\ln(x) + c$$

Utility Theory (cont'd)

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utility $\longrightarrow$ $u(x) = a\ln(x) + c$ *money amount*

parameters (person-specific)

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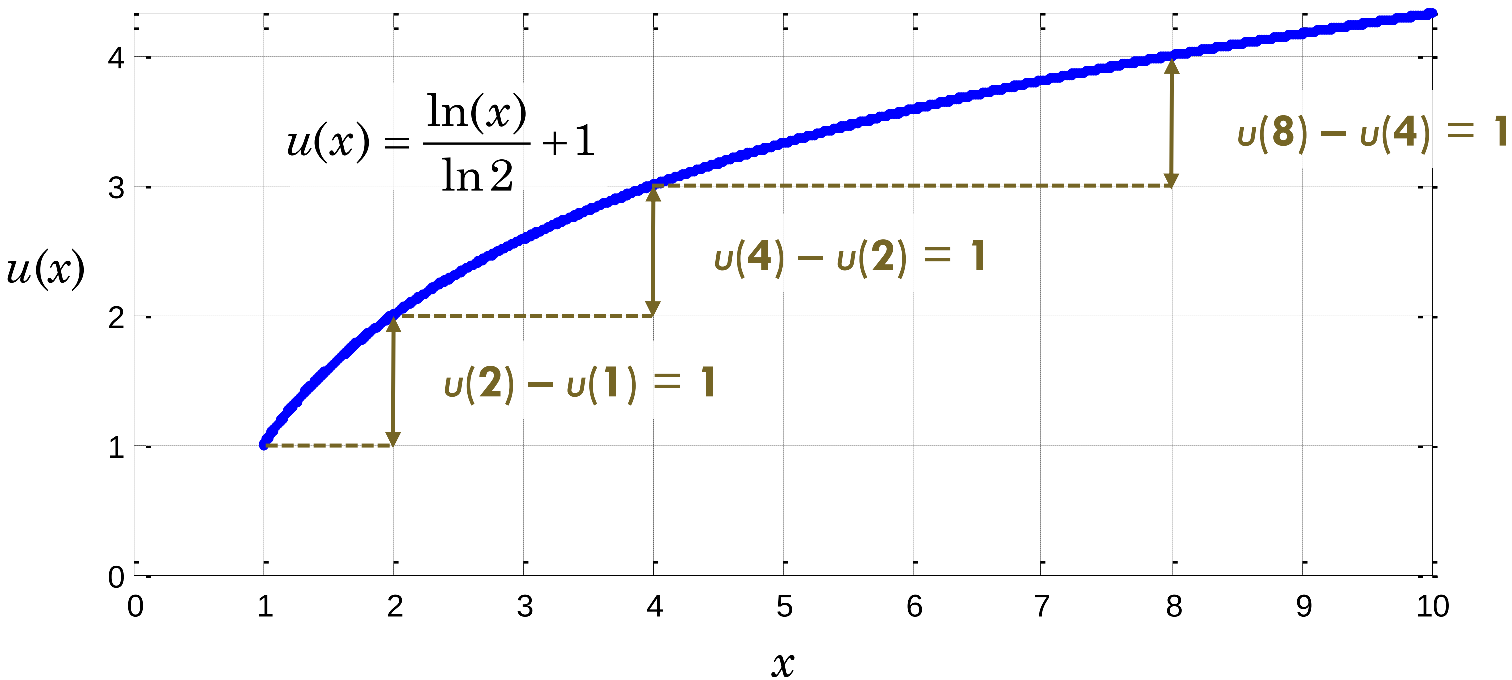

Bernoulli's Paradox Vs. Utility Theory

- if we use Bernoulli's logarithmic utility function, and watch *expected utility* instead of expected payoffs, Bernoulli's paradox ceases to be a paradox
- consider a utility function $u(x) = \ln(x)$

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- consider a utility function $u(x) = \ln(x)$; expected utility is

$$\mathrm{E}(u) = \sum_x u(x)\Pr(X = x) =$$

$$= \ln(1)\cdot\tfrac{1}{2} + \ln(2)\cdot\left(\tfrac{1}{2}\right)^2 + \ldots + \ln(2^n)\cdot\left(\tfrac{1}{2}\right)^{n+1} + \ldots =$$

(*T* + *H-T* + ... + *n*×*H-T* + ...)

$$= \sum_{n=0}^{\infty} \ln(2^n)\cdot\left(\tfrac{1}{2}\right)^{n+1} =$$

(only expressed as an infinite sum)

$$= \ln(2)\sum_{n=0}^{\infty} n\left(\tfrac{1}{2}\right)^{n+1} =$$

(remember: $\ln(2^n) = n\cdot\ln(2)$)

$$= \ln(2) = 0.69$$

(the sum equals 1 – *rather difficult to show*)

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$$= \ln(2) = 0.69$$ (the sum equals 1 – *rather difficult to show*)

→ a rational decision is to sell the ticket for any amount of money that yields greater utility than ln(2) → $\ln(2) < \ln(\textit{price})$ → e.g. for €2.5

Bernoulli's Paradox Vs. Utility Theory (cont'd)

- however, imagine we change the rules in the following way: if a tail first appears after n tosses, the payoff is $\exp(2^n)$
- the expected utility is:

$$\begin{aligned}
\mathrm{E}(u) &= \sum_x u(x)\Pr(X = x) = \\
&= \sum_{n=0}^{\infty} \ln\left[\exp(2^n)\right] \cdot \left(\tfrac{1}{2}\right)^n = \\
&= \sum_{n=0}^{\infty} 2^n \left(\tfrac{1}{2}\right)^n = \sum_{n=0}^{\infty} 1 = \infty
\end{aligned}$$

- note: for any kind of *unbounded* and *increasing* utility function, one can find a modified version of Bernoulli's lottery with infinite expected utility

Criticism of Bernoulli's Paradox

1. people simply do not believe in extremely improbable possibilities (however, they are willing to take part in real lotteries!)
2. the maximum payoff cannot be unlimited – "casino ruining" (no infinite sum = no problem)

payoff limit	expected payoff
€10	2
€100	3.5
€1,000	5
€1,000,000	10
€1,000,000,000,000	20
€1,000,000,000,000,000,000	30

3. utility cannot be unbounded, as there is a limited amount of scarce resources money can buy (?)

Properties of Typical Utility Functions

- typically, we assume a utility function $u(x)$ is a smooth function such that…
  1. **positivity**: $u(x) > 0$ for $x > 0$ (or, sometimes, for $x > 1$)
  2. **non-satiation**: $u'(x) > 0$
     - "the more money, the more utility"
     - sometimes, this assumption is made even stronger by assuming *unboundedness*
  3. **risk aversion**: $u''(x) < 0$
     - (for explanation, see next slide)
- logarithmic utility function: $u(x) = a \ln(x) + c, \quad a > 0, c > 0$
  - *positivity*: $u(x)$ crosses 0 at x between 0 and 1
  - *non-satiation*: $u'(x) = a/x > 0$ for $x > 0$
  - *risk aversion*: $u''(x) = -a/x^2 < 0$

Properties of Typical Utility Functions (cont'd)

Risk aversion

- risk aversion property merely states that $u(x)$ is *concave*
- to see why this results in risk aversion, consider the following situation:
  - Peter, whose utility function is

$$u(x) = \frac{\ln(x)}{\ln 2} + 1\,,$$

    was given a lottery ticket with the following lottery rules:
    - a fair coin is flipped:
      - *heads*: player wins \$2
      - *tails*: player wins \$8
    - the ticket itself can be sold back for \$5

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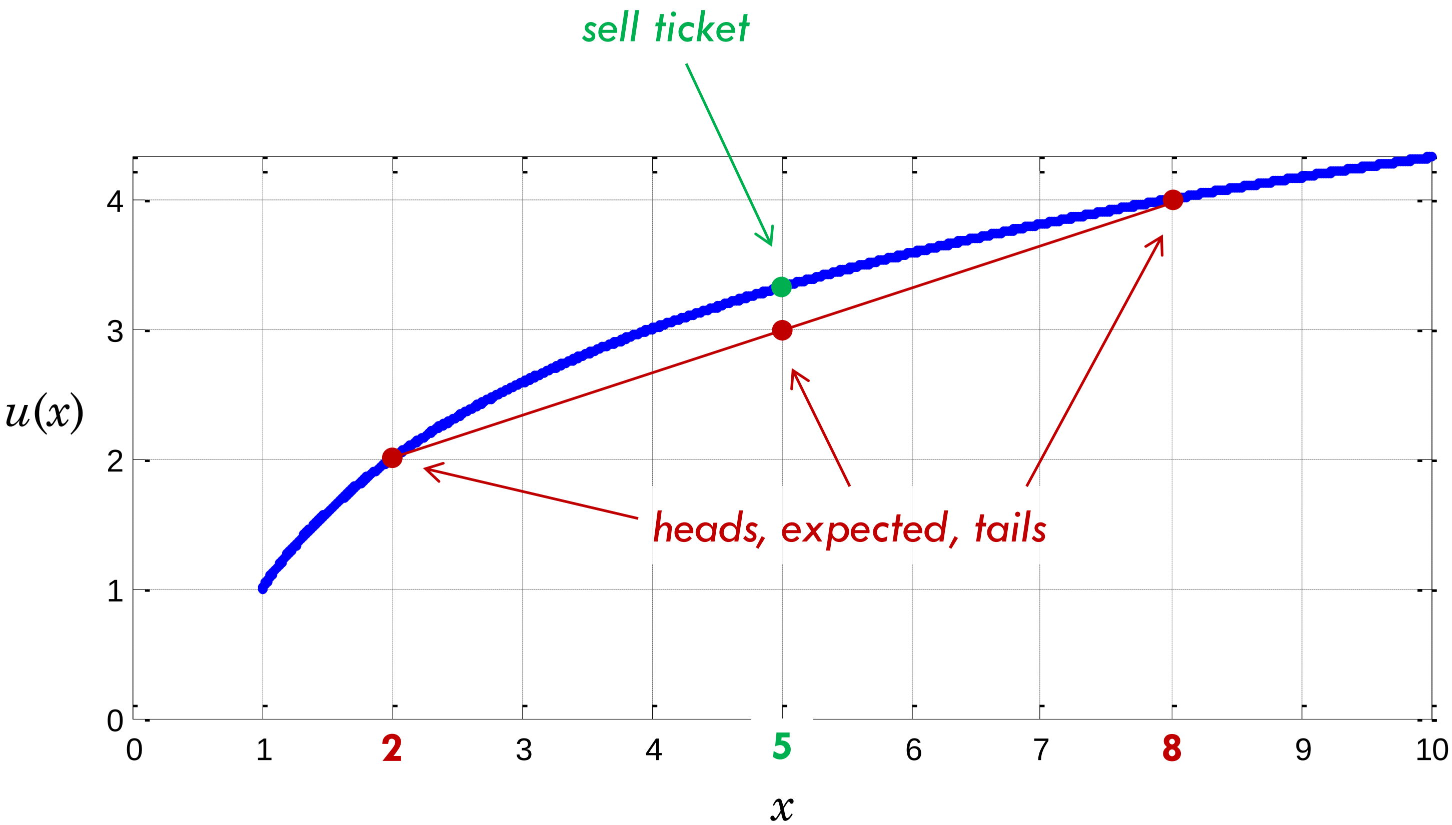
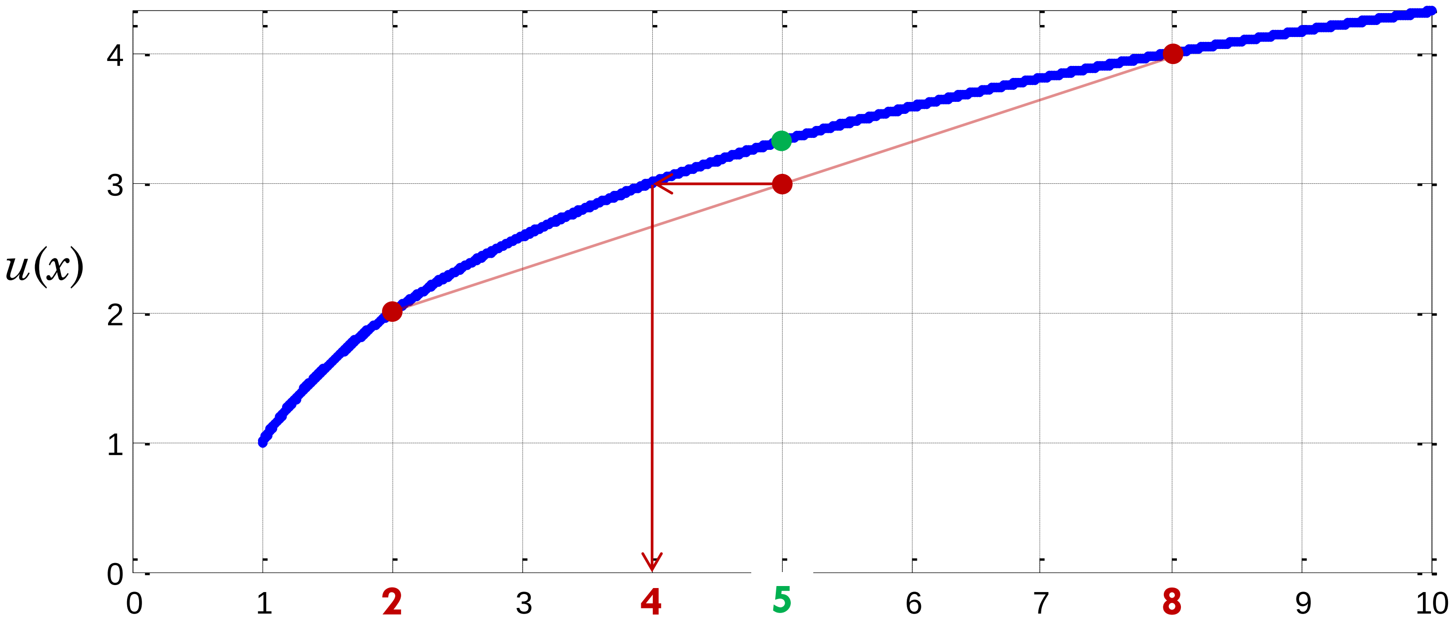
    was given a lottery ticket with the following lottery rules:
    - a fair coin is flipped:
      - *heads*: player wins \$2 ($u(2) = 2$)
      - *tails*: player wins \$8 ($u(8) = 4$)
    - the ticket itself can be sold back for \$5 ($u(5) = \mathbf{3.32}$)
  - if Peter doesn't sell the ticket:
    - expected payoff: $0.5{\times}2 + 0.5{\times}8 = \$5$ $\rightarrow$ fair lottery
    - expected utility: $0.5{\times}u(2) + 0.5{\times}u(8) = \mathbf{3}$ $\rightarrow$ Peter sells



sell ticket
heads, expected, tails
u(x)
x
0
1
2
3
4
0
1
2
3
4
5
6
7
8
9
10



certainty equivalent
to the expected payoff

Games against *p*-Intelligent Players

- real-life players do not often decide the way game theory suggests (i.e., game-theoretical result are not 100% *normative*)
- possible reasons:
  - different levels of information and/or decision skills
  - lack of time to analyze and decide
  - …
- mathematical model that counts in decision-making errors: *games with p-intelligent players*
  - **definition:** a player behaving with a probability of p like a normatively intelligent player and with a probability of $1-p$ like a random mechanism will be called a ***p*-intelligent player** ($p \in [0,1]$).
  - p = the degree of deviation from rationality:
    - $p = 0$ $\rightarrow$ a random mechanism
    - $p = 1$ $\rightarrow$ a completely rational player
  - *note*: your opponent's p needs to be estimated in advance!

p-Intelligent Players in Matrix Games

- consider the following matrix game
  - player 1 is normatively intelligent
  - player 2 is p-intelligent
  - the game's matrix is an $m \times n$ matrix $\boldsymbol{A} = (a_{ij})$
  - there exist NE strategies $\boldsymbol{x}^*$, $\boldsymbol{y}^*$
    - NE's may be *pure* or *mixed*, in either case the strategies $\boldsymbol{x}^*$, $\boldsymbol{y}^*$ will be expressed as vectors (for pure strategies, the vectors look something like $(0,1,0,0,0)^\top$)

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- by definition, player 2 plays a mixed strategy:
  - with a probability of p, he/she plays $\boldsymbol{y}^*$ (*rational*)
  - with a probability of $1-p$, he/she plays $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^\top$ (*random*)
  - the resulting mixed strategy is

$$\boldsymbol{s}(p) = p\,\boldsymbol{y}^* + (1-p)\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^\top$$

p-Intelligent Players in Matrix Games (cont'd)

- **optimal strategy for the intelligent player**: pick the row in $\boldsymbol{A}$ that maximizes the expected payoff, given that player 2 uses strategy $\boldsymbol{s}(p)$
- *mathematically*: find the maximum element in vector $\boldsymbol{A}\boldsymbol{s}(p)$

Example:

- NE: $\boldsymbol{x}^* = (1,0,0,0)^\top$, $\boldsymbol{y}^* = (0,1,0,0)^\top$
- $\boldsymbol{s}(p) = p\boldsymbol{y}^* + (1-p)\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)^\top =$

$$= p\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + (1-p)\begin{pmatrix}\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\\\frac{1}{4}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1-p\\1+3p\\1-p\\1-p\end{pmatrix}$$

1 \ 2	W	X	Y	Z
A	3	3	3	3
B	7	1	7	7
C	3	1	-1	2
D	8	0	8	8

What is the expected payoff of the first-row strategy with $p = 0.5$?

p-Intelligent Players in Matrix Games (cont'd)

- expected $\boldsymbol{A}\boldsymbol{s}(p)$ for different levels of p:

Row	$p = 0$	$p = 0.2$	$p = 0.4$	$p = 0.6$	$p = 0.8$	$p = 1$
A	3	3	3	3	3	3
B	5.5	4.6	3.7	2.8	1.9	1
C	1.25	1.20	1.15	1.10	1.05	1
D	6	4.8	3.6	2.4	1.2	0

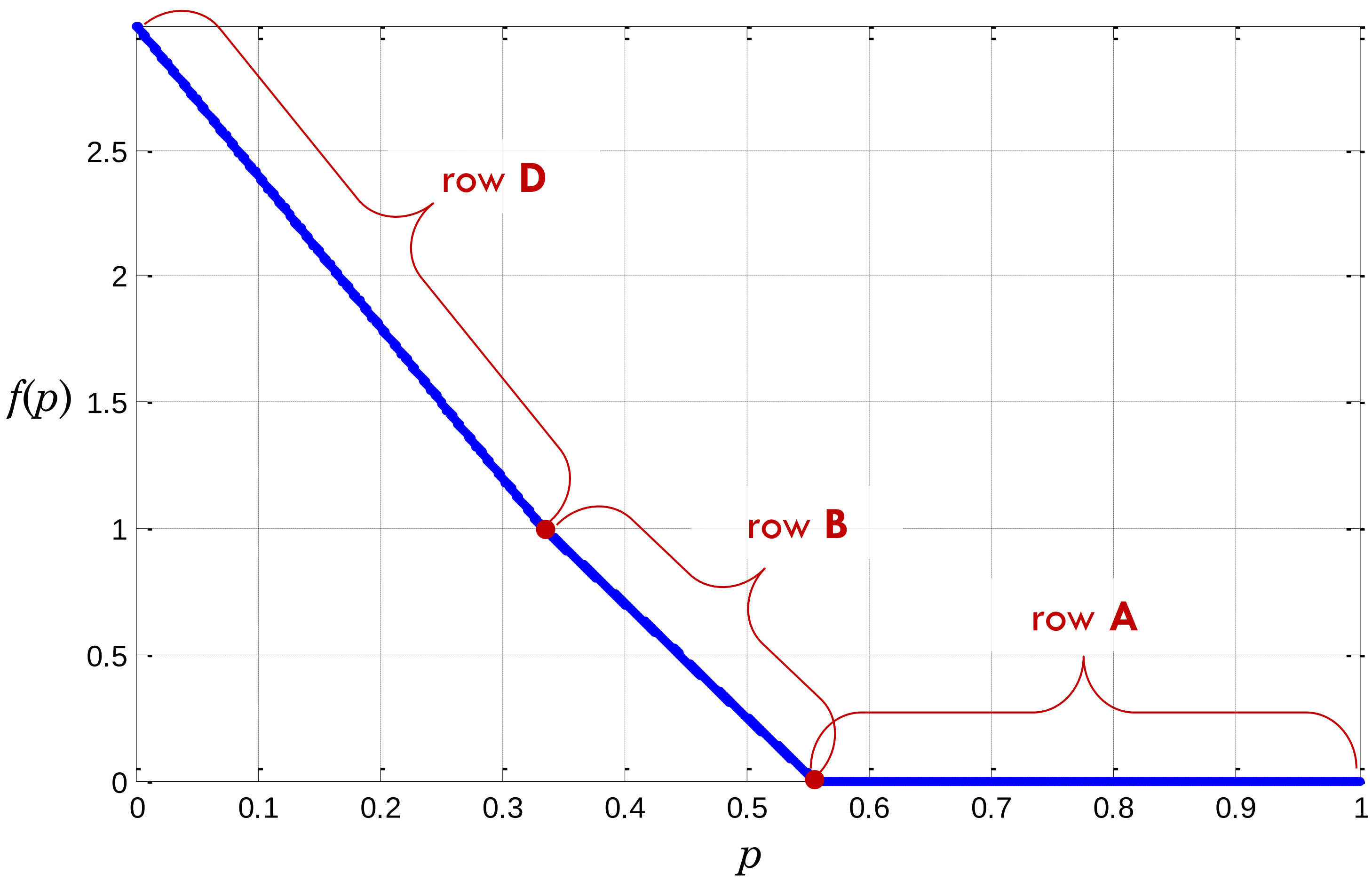
- depending on p, different rows can be optimal:
  - $p \in [0, 3/9] = [0, 0.33] \rightarrow$ row **D** is optimal
  - $p \in [3/9, 5/9] = [0.33, 0.56] \rightarrow$ row **B** is optimal
  - $p \in [5/9, 1] = [0.56, 1] \rightarrow$ row **A** is optimal

Excess Function

- excess function is a function that expresses the average additional player 1's profit due to his deviation from $\boldsymbol{x}^*$ (= NE strategy)

Row	$p = 0$	$p = 0.2$	$p = 0.4$	$p = 0.6$	$p = 0.8$	$p = 1$
A	3	3	3	**3**	**3**	**3**
B	5.5	4.6	**3.7**	2.8	1.9	1
C	1.25	1.20	1.15	1.10	1.05	1
D	**6**	**4.8**	3.6	2.4	1.2	0
excess	6–3 = 3	1.8	0.7	0	0	0

- *mathematically*: $$f(p) = \max[\boldsymbol{A}\boldsymbol{s}(p)] - \boldsymbol{x}^{*\top}\boldsymbol{A}\boldsymbol{s}(p)$$


row D
row B
row A
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$f(p)$
p

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