LECTURE 8:
NE's IN COMPLICATED GAMES, RISK AND UNCERTAINTY

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Games and Decisions

Finding Equilibria in Complicated Games

- consider an oligopoly with four players:
  - price function $p = f(x_1 + \ldots + x_4) = 100 - (x_1 + \ldots + x_4)$
  - capacities & costs:

$$X_1 = [0,40] \quad c_1(x_1) = 150 + 12x_1$$
$$X_2 = [0,20] \quad c_2(x_2) = x_2^2$$
$$X_3 = [0,30] \quad c_3(x_3) = 5 + 30x_3$$
$$X_4 = [0,10] \quad c_4(x_4) = \exp(x_4/2)$$

Questions:

Do you know how to find a NE combination of strategies in such an oligopoly?

Is it a much more difficult task than in case of the oligopolies we discussed in the last lesson?

What is the most difficult thing about it? Four players, or player 4's cost function?

Finding Equilibria in Complicated Games

- consider an oligopoly with four players:
  - price function $$p = f(x_1 + \ldots + x_4) = 100 - (x_1 + \ldots + x_4)$$
  - capacities & costs: $$X_1 = [0,40] \quad c_1(x_1) = 150 + 12x_1$$ $$X_2 = [0,20] \quad c_2(x_2) = x_2^2$$ $$X_3 = [0,30] \quad c_3(x_3) = 5 + 30x_3$$ $$X_4 = [0,10] \quad c_4(x_4) = \exp(x_4/2)$$

Questions:

Do you know how to find a NE combination of strategies in such an oligopoly?

Is it a much more difficult task than in case of the oligopolies we discussed in the last lesson?

What is the most difficult thing about it? Four players, or player 4's cost function?

- instead of solving the problem analytically, we can use a (fairly simple) simulation approach to find a NE: **fictitous game method**

Fictitious game method

- the procedure is as follows (typically realized as a computer simulation):
  1. players play the game *repetitively* (= in *rounds*)
  2. in the first round, players start with arbitrary strategies
  3. in all consecutive rounds, players assume their opponents will stick to their strategies from the previous round, and pick the best response to these strategies
  4. after a sufficient number of rounds, the combination of strategies should converge to a NE (if a NE exists)
- *notes*:
  - if in round r all players play the NE strategies, they will play NE strategies in round $r + 1$ as well (because NE = best response)
  - if there are no NE's, the fictitious game methods fails to converge
  - if there are multiple NE's, the fictitious game methods can only converge to one of them, depending on the starting strategies
    $\rightarrow$ multiple starting points should be tried out

Matlab implementation for oligopolies (available with comments on my website):

```
n = 4;
price = inline('100 - Q');
cost = {'150 + 12*q';
        'q^2';
        '5 + 30*q';
        'exp(q/2)'};
con = [40,20,30,10];
x = [5,5,5,5];
rounds = 60;
```

oligopoly parameters (price & cost functions, capacity constraints, initial strategies)

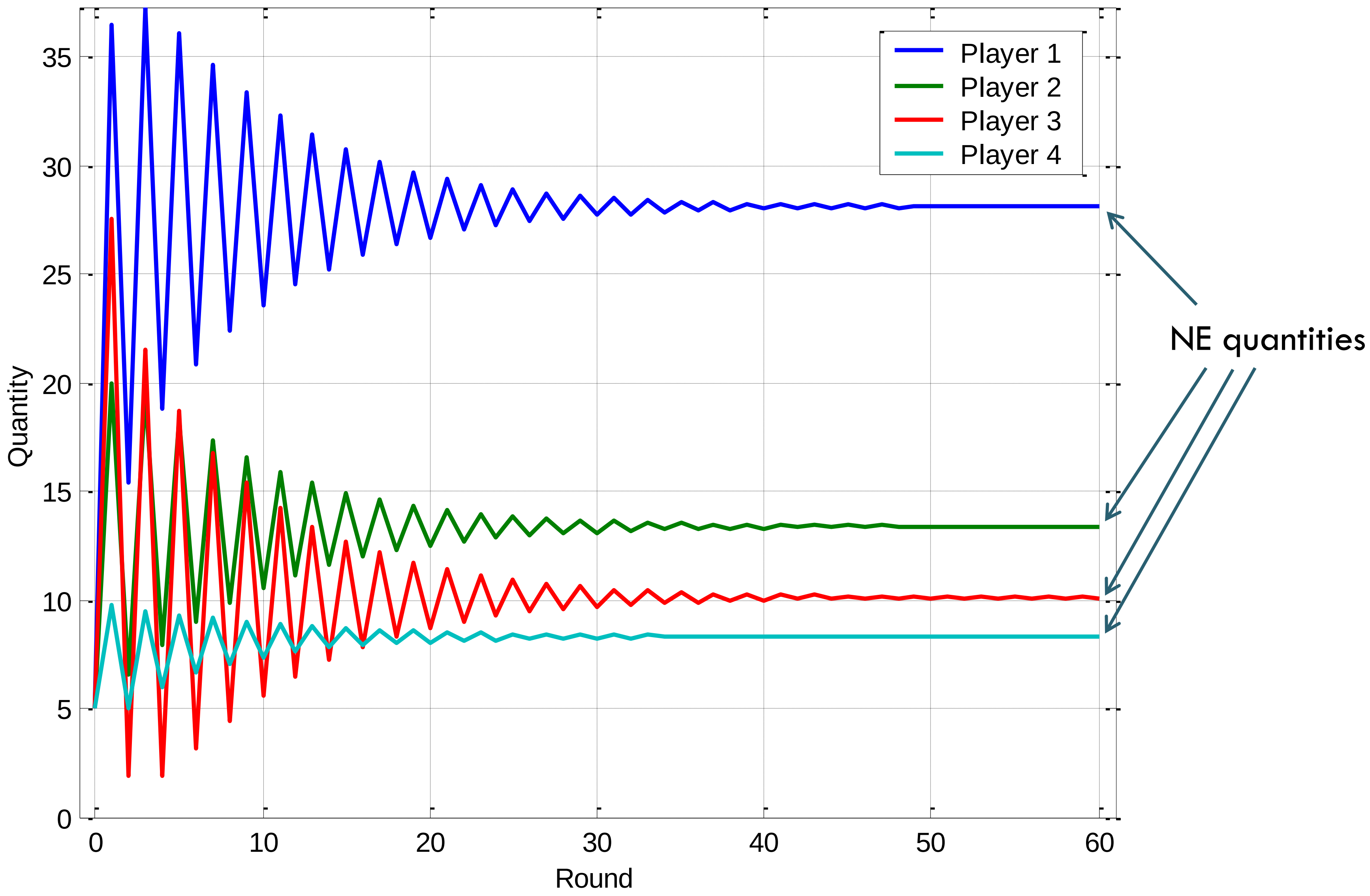
```
for r = 1:rounds
    x(r+1,:) = zeros(1,n);
    for i = 1:n
        Q = sum(x(r,:)) - x(r,i);
        c = inline(cost{i});
        x(r+1,i) = fminbnd(@profit,0,con(i),[],price,c,Q);
    end
end
```

simulation code

```
function prof = profit(q,price,cost,Q);
prof = q*price(Q+q) - cost(q);
prof = -prof;
```

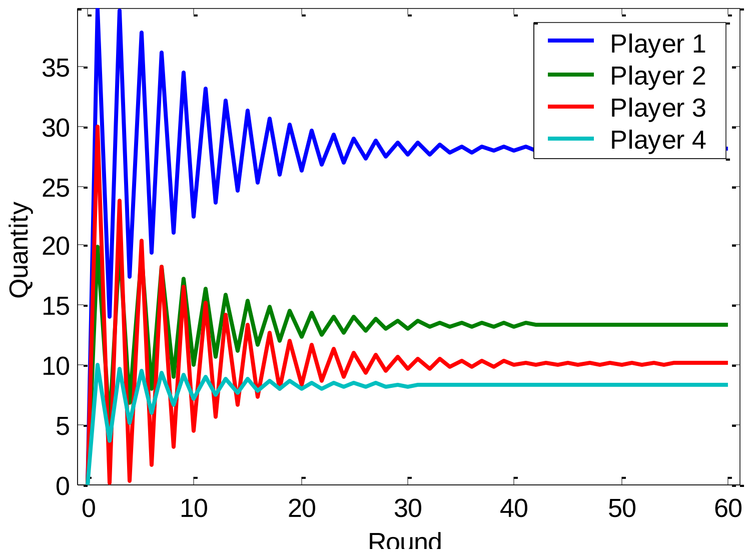
profit function definition

Simulation run

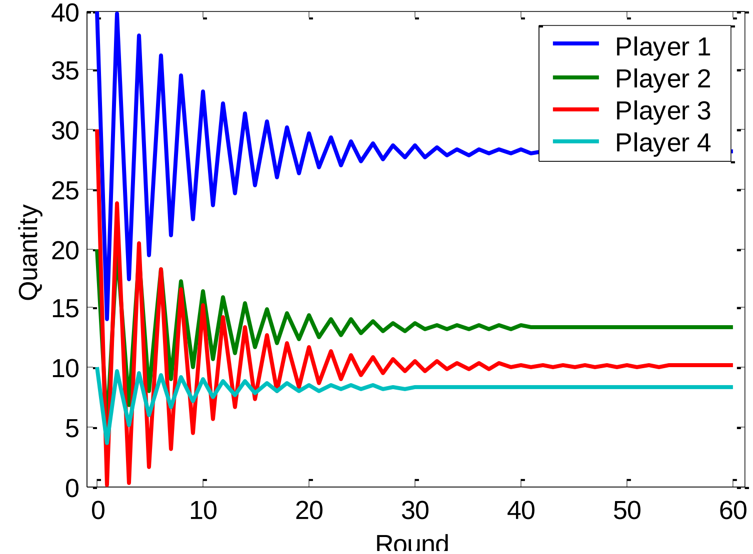


Different initial strategies (x_1, x_2, x_3, x_4):

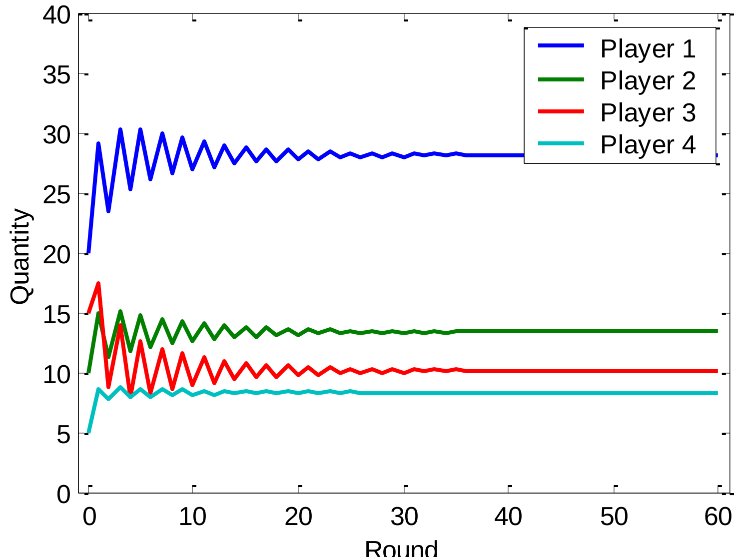
"Nothing": (0,0,0,0)



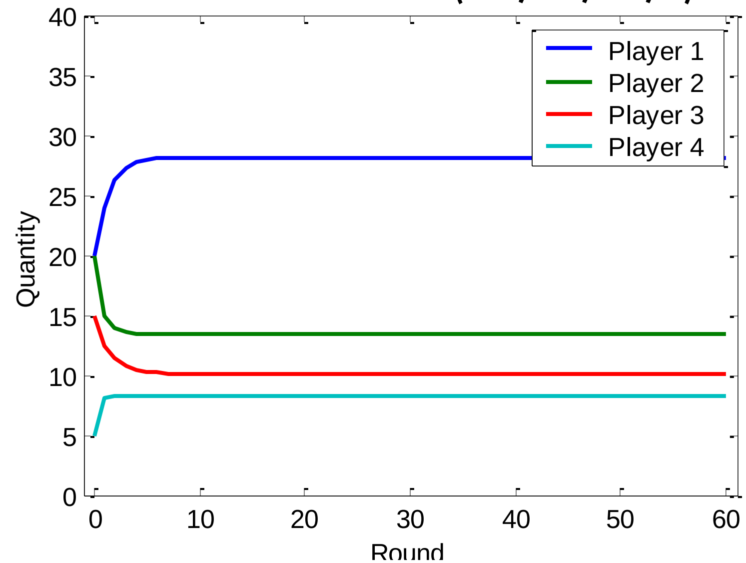
"Full Capacities": (40,20,30,10)



"Half Capacities": (20,10,15,5)

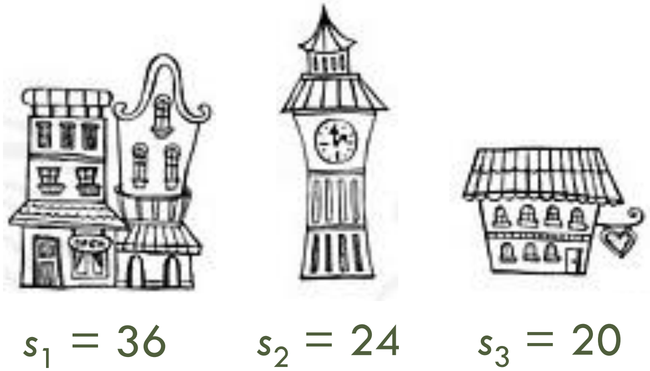


"Another One": (20,20,15,5)



Exercise 1: A Complicated Auction

- consider a multi-object sealed-bid auction from exercise 2, lecture 5
  - three objects, two players
  - object values: $s_1 = 36$, $s_2 = 24$, $s_3 = 20$
  - $I_1 = 20$, $I_2 = 10$
  - bids = multiples of 10
- Imagine the total investment amount changed to $I_1 = 40$, $I_2 = 30$



1. How many (pure) strategies does player 2 have?
2. How many (pure) strategies does player 1 have?
3. If we represent this auction as a bimatrix game, what will be the size of payoff matrices?

Exercise 1: A Complicated Auction (cont'd)

Binomial coefficient & combinations:

Consider a set of *n* objects. A *k*-combination is a subset made up by *k* distinct objects; i.e., it's a selection of *k* objects out of the total *n* objects. The number of *k*-combinations (or the number of ways one can select *k* out of *n* objects) is given by the *binomial coefficient* $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \times (n-1) \times \ldots \times (n-k+1)}{k \times (k-1) \times \ldots \times 1}$.

- *number of strategies of player 1*: how many ways can we divide a bundle of 4 objects (units of money) into three bundles (bids 1,2 and 3)?

Exercise 1: A Complicated Auction (cont'd)

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- *number of strategies of player 1*: how many ways can we divide a bundle of 4 objects (units of money) into three bundles (bids 1,2 and 3)?
  - put the objects in a line, place two "dividers" between them:

Exercise 1: A Complicated Auction (cont'd)

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- *number of strategies of player 1*: how many ways can we divide a bundle of 4 objects (units of money) into three bundles (bids 1,2 and 3)?
  - put the objects in a line, place two "dividers" between them:

$\rightarrow$ we choose 2 out of 6 positions for "dividers": $\binom{6}{2} = \frac{6 \times 5}{2 \times 1} = 15.$

Exercise 1: A Complicated Auction (cont'd)

- instead of calculating the payoff matrices, the *fictitious game method* can be applied
- in order to find out the best response to a given player 2's strategy, player 1 only needs to calculate a single column in his/her payoff matrix
- computer simulation is a bit more difficult than in case of oligopolies (see the Matlab implementation on my website), but still the program is rather short and simple

- *results*: for any combination of starting strategies, the simulation finds the following NE within 5 rounds:
  - $x^* = (2,1,1)$, $y^* = (2,1,0)$
  - $Z_1(x^*, y^*) = 2.5$, $Z_2(x^*, y^*) = 1.5$

Example: Multiple NE's

- consider the auction from exercise 2, lecture 5: there are two NE's, (**27**;**13**) is dominated by (**36**;**14**)

  → rational players will choose (**36**;**14**)
- despite this domination, fictitious game method may end up with *either* (**27**;**13**) *or* (**36**;**14**), depending on the starting point

Investor 2

Investor 1

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	16 ; 0	16 ; 14	16 ; 10
1,1,0	**27 ; 13**	33 ; 7	40 ; 10
1,0,1	23 ; 13	**36 ; 14**	31 ; 5
0,2,0	4 ; 26	4 ; 0	4 ; 10
0,1,1	24 ; 26	17 ; 7	19 ; 5
0,0,2	0 ; 26	0 ; 14	2 ; 0

$s_1 = 36$ $s_2 = 24$ $s_3 = 20$

Exercise 2: A Complicated Auction 2

- consider the following a multi-object sealed-bid auction with 4 objects and 4 players
  - total investments: $I_1 = 20$, $I_2 = 10$, $I_3 = 20$, $I_4 = 20$
  - bids = multiples of 10
  - object values:

$s_1 = 30$ $s_2 = 19$ $s_3 = 80$ $s_4 = 20$

1. Can this game be modelled as a bimatrix game?
2. How many different strategy profiles are there in this game?
3. Using the fictitious game method, we found the following NE strategies:

$$x_1{}^* = x_3{}^* = x_4{}^* = (0,0,20,0),\ \ x_2{}^* = (10,0,0,0).$$

   Calculate the expected payoffs.

Risk and Uncertainty

- many economic decisions made under risk and uncertainty
- risk and uncertainty can be modelled as a game with a non-rational opponent (random mechanism, “Nature”, “Fate”)
- with a non-rational opponent, none of the principles we discussed in previous lectures can be used
- **risk**: we know the possible outcomes (= opponent’s strategies), and the probabilities with which each of them may happen
- **uncertainty**: we know the possible outcomes, but *not* the probabilities

- *note: if we do not know the possible outcomes, it’s hard to produce any sort of a model* ☺

Decisions under Uncertainty: An Example

- Chemical Products Ltd. considers signing a contract to produce HIV testing sets for a hospital
- if they decide to sign a contract, they may choose to produce one of the following quantities: 2000 3000 4000 5000
- total production cost: \$20,000 25,000 30,000 35,000
- the price of a testing set depends on their reliability → destructive random sampling test before the sets are purchased
- resulting price:
  - $< 2\%$ give false results …… \$20 per testing set
  - $2 – 4\%$ give false results …… \$10 per testing set
  - $> 4\%$ give false results …… \$2 per testing set

  → *three possible outcomes, probabilities unknown*

- Should Chemical Products sign the contract? For what quantity?

Decisions under Uncertainty: An Example (cont'd)

- formally, we can treat the decision as a two-player game: "Chemical Products Vs. Nature"

→ player 1 (Chemical Products): 5 strategies 0, 2, 3, 4, 5 (in 1000s)
player 2 (Nature): 3 strategies < 2%, 2 – 4%, > 4%

- the payoffs (profits) of player 1 can be written down into a matrix (payoffs of player 2 are not important – player 2 is not rational)

Nature (% defective)

Chemical Products (quantity)

1 \ 2	< 2%	2 – 4%	> 4%
0	0	0	0
2,000	20	0	-16
3,000	35	5	-19
4,000	50	10	-22
5,000	65	15	-25

Uncertainty – Decision Making Principles

- uncertain results → no "best strategy"; several decision making principles suggested:
- **Laplace Principle**: suppose columns are picked with equal probabilities → select the row that maximizes the average (i.e., expected) profit
- **Minimax principle** (a.k.a. Murphy's Law Principle): suppose that "if anything can go wrong, it will" → pick the row that guarantees the maximum worst-case payoff
- **Principle of Maximin Regret**: decisions often judged ex post – as if the player could have known the opponent's strategy. Protect yourself against objections like "you could have been $50 better off, if you had picked strategy 2." Pick the row that guarantees the minimum worst-case *regret* (= the difference between the actual payoff and the maximum payoff in the given column)
- **Hurwicz Principle**: first, calculate the optimistic and pessimistic payoffs for each row. Then, pick the row that maximizes the weighted average of the two, where the weights express your inclination towards optimism/pessimism

Uncertainty – Decision Making Principles (cont'd)

- *mathematical description*: denote the $m \times n$ payoff matrix as $\boldsymbol{A} = (a_{ij})$; the optimal decision is to choose row i for which…
  - (*Laplace Principle*) … $\frac{1}{n}\sum_{j=1}^{n} a_{ij}$ is *maximal.*
  - (*Minimax Principle*) … $\min_j a_{ij}$ is *maximal.*
  - (*Pr. of Maximin Regret*) … $\max_j \left[(\max_k a_{kj}) - a_{ij} \right]$ is *minimal.*
  - (*Hurwicz Principle*) … $\alpha \cdot \max_j a_{ij} + (1-\alpha) \cdot \min_j a_{ij}$ is *maximal.*
    - $\alpha \in (0,1)$ is the inclination towards optimism

Decisions under Uncertainty: An Example (cont'd)

- *Laplace Principle*: find the highest row average

Nature (% defective)

Chemical Products (quantity)

1 \ 2	< 2%	2 – 4%	> 4%	$\frac{1}{3} \cdot \sum_i a_{ij}$
0	0	0	0	0
2,000	20	0	-16	4/3
3,000	35	5	-19	21/3
4,000	50	10	-22	38/3
5,000	65	15	-25	**55/3**

→ according to the Laplace Principle, Chemical Products should sign a contract for 5,000 testing sets

Decisions under Uncertainty: An Example (cont'd)

- *Minimax Principle*: find the highest row minimum

Nature (% defective)

Chemical Products (quantity)

1 \ 2	< 2%	2 – 4%	> 4%	$\min_j a_{ij}$
0	**0**	**0**	**0**	**0**
2,000	20	0	**-16**	-16
3,000	35	5	**-19**	-19
4,000	50	10	**-22**	-22
5,000	65	15	**-25**	-25

→ according to the Minimal Principle, Chemical Products should not sign a contract

Decisions under Uncertainty: An Example (cont'd)

- *Principle of Maximin Regret*:
  1. find the **matrix of regrets** (subtract entries from column maxima)
  2. find the *lowest row maximum*

1 \ 2	< 2%	2–4%	> 4%
0	0	0	0
2,000	20	0	-16
3,000	35	5	-19
4,000	50	10	-22
5,000	65	15	-25

1 \ 2	< 2%	2–4%	> 4%	max
0	**65**	15	0	65
2,000	**45**	15	16	45
3,000	**30**	10	19	30
4,000	15	5	**22**	**22**
5,000	0	0	**25**	25

→ according to the Principle of Maximin Regret, Chemical Products should sign a contract for 4,000 testing sets

Decisions under Uncertainty: An Example (cont'd)

- *Hurwicz Principle*:
  1. select your level of optimism α (for now, let $\alpha = 0.75$)
  2. find row maxima (*max*) and minima (*min*)
  3. pick the row that maximizes $\alpha \cdot max + (1 - \alpha) \cdot min$

1 \ 2	< 2%	2 – 4%	> 4%
0	0	0	0
2,000	20	0	-16
3,000	35	5	-19
4,000	50	10	-22
5,000	65	15	-25

max	*min*
0	0
20	-16
35	-19
50	-22
65	-25

0.75×*max* – 0.25×*min*
0
11
21.5
32
42.5

→ according to Hurwicz Principle with $\alpha = 0.75$, Chemical Products should sign a contract for 5,000 testing sets

Lecture 8:

NE's in Complicated Games, Risk and Uncertainty

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Games and Decisions