Lecture 7:
Collusive Oligopoly (cont’d), Coalition Games

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Games and Decisions
Principle of Group Stability

- **imputation**: a potential final distribution of payoffs to all players \((a_1, a_2, \ldots, a_N)\)

- A coalition of 2 players is formed only if the total profit can be distributed so that both are better off: \(v(1,2) \geq v(1) + v(2)\), or, in other words, there exist \(a_1, a_2\) such that

\[
\begin{align*}
    a_1 + a_2 &= v(1,2), \\
    a_1 &\geq v(1), \\
    a_2 &\geq v(2).
\end{align*}
\]
Principle of Group Stability

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  \[
  a_1 + a_2 = v(1,2),
  \]
  \[
  a_1 \geq v(1),
  \]
  \[
  a_2 \geq v(2).
  \]

- similarly, a coalition of \(m\) players can be formed only if it pays for *all its subcoalitions* to take part

  → **principle of group stability**: coalition \(K\) can be formed only if there exists an imputation that satisfies

  \[
  \sum_{i \in K} a_i = v(K),
  \]
  \[
  \sum_{i \in L} a_i \geq v(L) \quad \text{for all subcoalitions } L \subset K.
  \]
Exercise 1: Group Stability

- Consider an oligopoly with three firms and the following characteristic function

\[
v(\emptyset) = 0, \quad v(1,2) = 5.5, \\
v(1) = 2, \quad v(1,3) = 4, \\
v(2) = 3, \quad v(2,3) = 5.5, \\
v(3) = 2.5, \quad v(1,2,3) = 8.
\]

- Which of the four multiplayer coalitions are stable?
  - Write down the stability conditions explicitly for each of the four coalitions:

\[
\sum_{i \in K} a_i = v(K), \quad \sum_{i \in L} a_i \geq v(L) \quad \text{for all subcoalitions } L \subset K
\]

- Find feasible outcomes \(a_i\) for the stable coalitions, find colliding inequalities for the unstable ones.
Core of the Oligopoly

- a straightforward extension of the core of a cooperative bimatrix game:
- definition: a set of all imputations that satisfy
  \[ \sum_{i=1}^{N} a_i = v(Q), \]
  \[ \sum_{i \in K} a_i \geq v(K) \quad \text{for all coalitions } K. \]

- i.e., core is the set of all imputations that satisfy the principles of:
  - **group stability** for the grand coalition
  - **collective rationality** – maximum profit is generated
    → this assumes that the grand coalition generates the maximum total profit
  - sometimes defined in a different way for the case where the grand coalition doesn’t generate the total profit (complicated)

*note: there’s always a winning coalition, one that is stable and generates the maximum profit amongst the stable coalitions*
Exercise 2: Core of the Oligopoly

Consider an oligopoly with three firms and the following characteristic function

\[ v(\emptyset) = 0, \quad v(1,2) = 5.5, \]
\[ v(1) = 2, \quad v(1,3) = 4, \]
\[ v(2) = 3, \quad v(2,3) = 5.5, \]
\[ v(3) = 2.5, \quad v(1,2,3) = x. \]

How many imputations are there in the core of the game, given that

a) \( x = 10 \)?

b) \( x = 8 \)?

c) \( x = 7.5 \)?

Note: if there’s no imputation in the core, there’s an empty core.
consider an oligopoly with three firms:

- price function
  \[ p = f(x_1 + x_2 + x_3) = 6 - \frac{1}{2}(x_1 + x_2 + x_3) \]

- capacities & costs:
  \[ X_1 = [0, 6] \quad c_1(x_1) = \frac{1}{2} x_1 + 3 \]
  \[ X_2 = [0, 3] \quad c_2(x_2) = \frac{3}{4} x_2 + 2 \]
  \[ X_3 = [0, 2] \quad c_3(x_3) = \frac{5}{2} x_3 + 1 \]

in the competitive oligopoly setting, the NE is:

- \( x_1^* = 4, \ x_2^* = 3, \ x_3^* = 0. \)
- \( \pi_1^* = 5, \ \pi_2^* = 3.25, \ \pi_3^* = -1. \)
- 7 units sold at 2.5

equilibrium characteristic function:

\[ v(1,2,3) = 9.125, \quad x_1 = 5.5, \ x_2 = 0, \ x_3 = 0, \quad 5.5 \text{ units sold at 3.25} \]
\[ v(1,2) = 10.125, \quad x_1 = 5.5, \ x_2 = 0, \ x_3^* = 0, \quad 5.5 \text{ units sold at 3.25} \]
on itself, firm 3 always makes a loss → typically, leaves the market

blocking effect: imagine there are two scenarios

1. firm 3 leaves, firms 1 and 2 collude
   - firms 1,2 choose the output of (1,2) coalition, 5.5 units sold at 3.25
2. firm 3 is subsidized, remains in the market, creates competitive environment
   - firms 1,2 choose the NE output, 7 units sold at 2.5

- the difference in consumer surplus is at least
  \[ 5.5 \times (3.25 - 2.5) = 4.125 \]

- a subsidy of 1 is enough to keep firm 3 on the market
  → subsidizing firm 3 yields greater consumer surplus, even if the subsidy is paid by the consumers (the increase in consumer surplus is at least \[ 4.125 - 1 = 3.125 \])

- blocking effect subsidies are a form of a state regulation
a measure of players’ negotiating power in making coalitions
introduced by Lloyd Shapley in 1953
the formula is quite nasty, but the idea is quite simple, and so is the calculation for small \(N\)
imagine that the grand coalition is formed in such a way that players come in random order and gradually form the grand coalition (each newcomer joining the existing coalition)
the contribution of player \(i\) joining coalition \(K\) is defined as
\[
v(K \cup \{i\}) - v(K).
\]
the Shapley value of player \(i\) is the average \(i\)’s contribution, the Shapley value is a vector of such average contributions for all players. We are averaging across all possible ways the grand coalition can be formed, i.e. across all possible orderings of players
as the players come at random, each ordering of the players is equally likely; therefore, the result can be viewed as the expected contribution of player \(i\)
Example:

- consider the oligopoly from exercise 1:

<table>
<thead>
<tr>
<th>order</th>
<th>contribution of player 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3</td>
<td>v(1) − v(∅) = 2 − 0 = 2</td>
</tr>
<tr>
<td>1 3 2</td>
<td>v(1) − v(∅) = 2 − 0 = 2</td>
</tr>
<tr>
<td>2 1 3</td>
<td>v(1,2) − v(2) = 5.5 − 3 = 2.5</td>
</tr>
<tr>
<td>3 1 2</td>
<td>v(1,3) − v(3) = 4 − 2.5 = 1.5</td>
</tr>
<tr>
<td>2 3 1</td>
<td>v(1,2,3) − v(2,3) = 8 − 5.5 = 2.5</td>
</tr>
<tr>
<td>3 2 1</td>
<td>v(1,2,3) − v(2,3) = 8 − 5.5 = 2.5</td>
</tr>
</tbody>
</table>

Shapley value of player 1 = \(\sum / 6 = 13/6\)
Mathematically,

Shapley value of $i = \sum_{K \subseteq Q \setminus \{i\}} \frac{|K|!(N - |K| - 1)!}{N!} \left( v(K \cup \{i\}) - v(K) \right)

$|K|$ positions, $|K|!$ possible orders
$N - |K| - 1$ positions, $(N - |K| - 1)!$ possible orders

N positions, $N!$ possible orders
Exercise 3: Shapley values

- again, consider the oligopoly from exercise 1
- calculate the Shapley values of player 2 and 3

\[
\begin{align*}
  v(\emptyset) &= 0, & v(1,2) &= 5.5, \\
  v(1) &= 2, & v(1,3) &= 4, \\
  v(2) &= 3, & v(2,3) &= 5.5, \\
  v(3) &= 2.5, & v(1,2,3) &= 8.
\end{align*}
\]
the approach we used for collusive oligopolies can easily be extended to a wider framework of coalition games
deal with cooperative conflicts
typically modelled as games in characteristic function form

with coalition games, the typical task is to...
1. ... express the characteristic function explicitly
   sometimes, we have only a verbal description (rule) for the coalitions’ payoffs
2. ... find the core of the game (defined as in case of oligopolies)
3. ... calculate the Shapley value

Question:
Consider a coalition game with $n$ players. How many different coalitions can player 1 join?
(Remember: we treat the empty and single-member groups as coalitions, too)
Exercise 4: Miners

- consider a group of $n$ miners who have discovered large bars of gold
- two miners can carry one piece of gold, so the payoff of coalition $K$ is

$$v(K) = \begin{cases} 
\frac{|K|}{2}, & \text{if } |K| \text{ is even,} \\
\frac{(|K|-1)}{2}, & \text{if } |K| \text{ is odd,}
\end{cases}$$

where $|K|$ denotes the number of members of $K$.

- what is the core of the game? Assume that...
  - $n$ is even
  - $n$ is odd
- what’s the Shapley value of a miner in the game?
Core of the game:

- \( n \) is even
  - \( n = 2 \): the core is made up by all couples of non-negative \( a_1, a_2 \) with the total of 1 (\( \rightarrow \) infinite number of imputations)
    - a “fair” imputation: \( a_1 = a_2 = \frac{1}{2} \)
  - \( n > 2 \): core = a single imputation \( a_1 = a_2 = \ldots = a_n = \frac{1}{2} \)
    - first of all, if an imputation is in the core, it has to satisfy
      \[
      a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i = v(Q) = \frac{n}{2}
      \]
    - second, the stability conditions have to hold for all subcoalitions of the grand coalition (how many subcoalitions exist?)
    - we don’t have to write down all the conditions, as most of them are quite similar – players have identical conditions
stability conditions for the pairs of player 1 + player i:

\[
\begin{align*}
\alpha_1 + \alpha_2 & \geq 1 \\
\alpha_1 + \alpha_3 & \geq 1 \\
& \quad \vdots \\
\alpha_1 + \alpha_n & \geq 1
\end{align*}
\]

\[
(n-1)\alpha_1 + \alpha_2 + \ldots + \alpha_n \geq n-1
\]

\[
(n-2)\alpha_1 + \sum_{i=2}^{n} \alpha_i \geq n-1
\]

\[
(n-2)\alpha_1 + v(Q) \geq n-1
\]

\[
\alpha_1 \geq \frac{n-1-v(Q)}{n-2} = \frac{n-1-\frac{n}{2}}{n-2} = \frac{1}{2}
\]

→ the same holds for players 2, 3, ..., n → all players get \( \frac{1}{2} \)
Exercise 4: Miners

- $n$ is odd: the core is empty (!)
  - using the same approach as for even number of players, we obtain:
    
    $$(n - 2)a_i + v(Q) \geq n - 1$$

- now $v(Q) = (n - 1)/2$, which yields
  
  $$a_i \geq \frac{1}{n - 2} \cdot \frac{n - 1}{2}$$

- if all players receive this amount, the total is
  
  $$a_1 + a_2 + \ldots a_n = \frac{n}{n - 2} \cdot \frac{n - 1}{2} = \frac{n}{n - 2} \cdot v(Q)$$

- an imputation that satisfies both $a_1 + a_2 + \ldots a_n = v(Q)$ and the stability conditions doesn’t exist
Exercise 5: Shoes

☐ for the moment ignore shoe sizes: a pair consists of a left and a right shoe, which can then be sold for €10 (a single shoe on itself is worthless)

☐ consider a coalition game with 2001 players:
  ☐ 1000 of them have 1 left shoe
  ☐ 1001 of them have 1 right shoe

1. Would you prefer to be a right- or left shoe owner in this game?
   ☐ what ratio of profit shares would you expect for a pair of right- and left shoe owners who combine their shoes and sell the pair?

2. Find the characteristic function for a coalition of \( n \) left-shoe owners and \( m \) right-shoe owners

3. Find the core of the game
   ☐ Is the core empty? If not, are there multiple imputations in the core, or is there just one?
characteristic function for a coalition of $n$ left-shoe owners and $m$ right-shoe owners:

$$v(K) = 10 \times \min(m,n)$$

grand coalition: $v(Q) = 10,000$

stability conditions for pairs: each pair of a right- and left shoe owner must obtain at least €10 (no requirements for other kinds of pairs)

this is only possible when all left shoe owners get €10, right shoe owners get nothing (!)

→ criticism of the core concept
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