# Lecture 7: Collusive Oligopoly (cont'd), Coalition Games

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Games and Decisions

# Principle of Group Stability



- **imputation**: a potential final distribution of payoffs to all players ($a_1$, $a_2,\ldots,a_N$)
- a coalition of 2 players is formed only if the total profit can be distributed so that both are better off: $v(1,2) \geq v(1) + v(2)$ , or, in other words, there exist $a_1, a_2$ such that

$$
\begin{aligned}
a_1 + a_2 &= v(1,2), \\
a_1 &\geq v(1), \\
a_2 &\geq v(2).
\end{aligned}
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$$\begin{aligned} a_1 + a_2 &= v(1,2), \\ a_1 &\geq v(1), \\ a_2 &\geq v(2). \end{aligned}$$

- similarly, a coalition of $m$ players can be formed only if it pays for *all its subcoalitions* to take part

  → **principle of group stability**: coalition $K$ can be formed only if there exists an imputation that satisfies

$$\sum_{i \in K} a_i = v(K),$$

$$\sum_{i \in L} a_i \geq v(L) \quad \text{for all subcoalitions } L \subset K.$$

# Exercise 1: Group Stability



- Consider an oligopoly with three firms and the following characteristic function

$$\begin{aligned} v(\emptyset) &= 0, & v(1,2) &= 5.5, \\ v(1) &= 2, & v(1,3) &= 4, \\ v(2) &= 3, & v(2,3) &= 5.5, \\ v(3) &= 2.5, & v(1,2,3) &= 8. \end{aligned}$$

- which of the four multiplayer coalitions are *stable*?
  - write down the stability conditions explicitly for each of the four coalitions:

$$\sum_{i \in K} a_i = v(K), \quad \sum_{i \in L} a_i \geq v(L) \quad \text{for all subcoalitions } L \subset K$$

  - find feasible outcomes $a_i$ for the stable coalitions, find colliding inequalities for the unstable ones

# Core of the Oligopoly



- a straightforward extension of the core of a cooperative bimatrix game:
- definition: a set of all imputations that satisfy

$$\sum_{i=1}^{N} a_i = v(Q),$$
$$\sum_{i \in K} a_i \geq v(K) \quad \text{for all coalitions } K.$$

- i.e., core is the set of all imputations that satisfy the principles of:
  - **group stability** for the grand coalition
  - **collective rationality** – maximum profit is generated

    → this assumes that the grand coalition generates the maximum total profit
    - sometimes defined in a different way for the case where the grand coalition doesn't generate the total profit (complicated)
      (*note: there's always a* winning coalition, *one that is stable and generates the maximum profit amongst the stable coalitions*)

# Exercise 2: Core of the Oligopoly



- Consider an oligopoly with three firms and the following characteristic function

$$
\begin{aligned}
v(\emptyset) &= 0, & v(1,2) &= 5.5, \\
v(1) &= 2, & v(1,3) &= 4, \\
v(2) &= 3, & v(2,3) &= 5.5, \\
v(3) &= 2.5, & v(1,2,3) &= x.
\end{aligned}
$$

- How many imputations are there in the core of the game, given that
  *a)* $x = 10$ ?
  *b)* $x = 8$ ?
  *c)* $x = 7.5$ ?

*Note: if there's no imputation in the core, there's an* **empty core**.

# Blocking effect



- consider an oligopoly with three firms:
  - price function $$p = f(x_1 + x_2 + x_3) = 6 - \frac{1}{2}(x_1 + x_2 + x_3)$$
  - capacities & costs:
$$X_1 = [0,6] \quad c_1(x_1) = \frac{1}{2}x_1 + 3$$
$$X_2 = [0,3] \quad c_2(x_2) = \frac{3}{4}x_2 + 2$$
$$X_3 = [0,2] \quad c_3(x_3) = \frac{5}{2}x_3 + 1$$
- in the competitive oligopoly setting, the NE is:
  - $x_1^* = 4$, $x_2^* = 3$, $x_3^* = 0$.
  - $\pi_1^* = 5$, $\pi_2^* = 3.25$, $\pi_3^* = -1$.
  - 7 units sold at 2.5
- equilibrium characteristic function:

$v(1,2,3) = 9.125$, $x_1 = 5.5$, $x_2 = 0$, $x_3 = 0$, 5.5 units sold at 3.25

$v(1,2) = 10.125$, $x_1 = 5.5$, $x_2 = 0$, $x_3^* = 0$, 5.5 units sold at 3.25

# Blocking effect (cont'd)



- on itself, firm 3 always makes a loss → typically, leaves the market
- **blocking effect**: imagine there are two scenarios
  1. firm 3 leaves, firms 1 and 2 *collude*
     - firms 1,2 choose the output of (1,2) coalition, 5.5 units sold at 3.25
  2. firm 3 is subsidized, remains in the market, creates *competitive* environment
     - firms 1,2 choose the NE output, 7 units sold at 2.5
  - the difference in consumer surplus is at least

$$5.5 \times (3.25 - 2.5) = 4.125$$

  - a subsidy of 1 is enough to keep firm 3 on the market

  → subsidizing firm 3 yields greater consumer surplus, even if the subsidy is paid by the consumers (the increase in consumer surplus is at least $4.125 - 1 = 3.125$)

- blocking effect subsidies are a form of a state regulation

# Shapley Value



- a measure of players' negotiating power in making coalitions
- introduced by Lloyd Shapley in 1953
- the formula is quite nasty, but the idea is quite simple, and so is the calculation for small $N$
- imagine that the grand coalition is formed in such a way that players come in random order and gradually form the grand coalition (each newcomer joining the existing coalition)
- the **contribution** of player $i$ joining coalition $K$ is defined as

$$v(K \cup \{i\}) - v(K).$$

- the **Shapley value of player** $i$ is the average $i$'s contribution, the **Shapley value** is a vector of such average contributions for all players. We are averaging across all possible ways the grand coalition can be formed, i.e. across all possible orderings of players
- as the players come at random, each ordering of the players is equally likely; therefore, the result can be viewed as the *expected contribution* of player $i$

# Shapley Value



**Example**:

- consider the oligopoly from exercise 1:

$$v(\emptyset) = 0, \quad v(1,2) = 5.5,$$
$$v(1) = 2, \quad v(1,3) = 4,$$
$$v(2) = 3, \quad v(2,3) = 5.5,$$
$$v(3) = 2.5, \quad v(1,2,3) = 8.$$

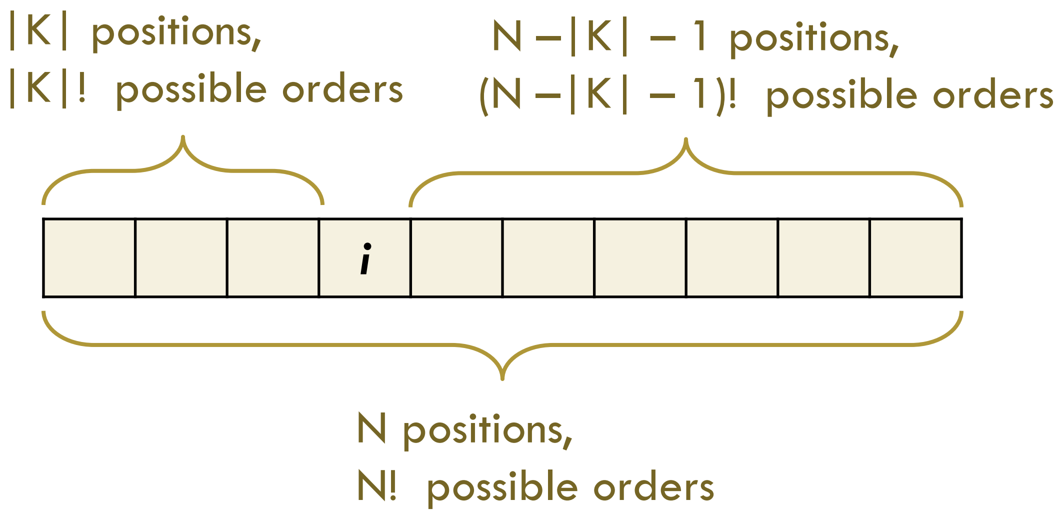
| order | contribution of player 1 | | |
|---|---|---|---|
| **1** 2 3 | v(1) – v (Ø) = | 2 – 0 = | **2** |
| **1** 3 2 | v(1) – v (Ø) = | 2 – 0 = | **2** |
| 2 **1** 3 | v(1,2) – v (2) = | 5.5 – 3 = | **2.5** |
| 3 **1** 2 | v(1,3) – v (3) = | 4 – 2.5 = | **1.5** |
| 2 3 **1** | v(1,2,3) – v (2,3) = | 8 – 5.5 = | **2.5** |
| 3 2 **1** | v(1,2,3) – v (2,3) = | 8 – 5.5 = | **2.5** |
| | **Shapley value of player 1** | **= ∑ / 6 =** | **13/6** |

# Shapley Value



- Mathematically,

$$\text{Shapley value } of\ i = \sum_{K \subseteq Q \setminus \{i\}} \frac{|K|!(N-|K|-1)!}{N!}\Big(v(K \cup \{i\}) - v(K)\Big)$$



# Exercise 3: Shapley values



- again, consider the oligopoly from exercise 1
- calculate the Shapley values of player 2 and 3

$$
\begin{aligned}
v(\emptyset) &= 0, & v(1,2) &= 5.5, \\
v(1) &= 2, & v(1,3) &= 4, \\
v(2) &= 3, & v(2,3) &= 5.5, \\
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\end{aligned}
$$

# Coalition Games



- the approach we used for collusive oligopolies can easily be extended to a wider framework of **coalition games**
  - deal with *cooperative conflicts*
  - typically modelled as games in *characteristic function form*
- with coalition games, the typical task is to…
  1. … express the *characteristic function* explicitly
     - sometimes, we have only a verbal description (rule) for the coalitions' payoffs
  2. … find the *core of the game* (defined as in case of oligopolies)
  3. … calculate the *Shapley value*

**Question:**

Consider a coalition game with *n* players. How many different coalitions can player 1 join?

*(Remember: we treat the empty and single-member groups as coalitions, too)*

# Exercise 4: Miners



- consider a group of $n$ miners who have discovered large bars of gold
- two miners can carry one piece of gold, so the payoff of coalition $K$ is

$$v(K) = \begin{cases} |K|/2, & \text{if } |K| \text{ is even,} \\ \left(|K|-1\right)/2, & \text{if } |K| \text{ is odd,} \end{cases}$$

where $|K|$ denotes the number of members of $K$.

- what is the core of the game? Assume that…
  - $n$ is even
  - $n$ is odd
- what's the Shapley value of a miner in the game?

# Exercise 4: Miners (cont'd)



*Core of the game*:

- $n$ is even
  - $n = 2$: the core is made up by all couples of non-negative $a_1$, $a_2$ with the total of 1 (→ *infinite* number of imputations)
    - a "fair" imputation: $a_1 = a_2 = ½$
  - $n > 2$: core = a single imputation $a_1 = a_2 = … = a_n = ½$
    - first of all, if an imputation is in the core, it has to satisfy

$$a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i = v(Q) = \frac{n}{2}$$

    - second, the stability conditions have to hold for all subcoalitions of the grand coalition (how many subcoalitions exist?)
    - we don't have to write down all the conditions, as most of them are quite similar – players have identical conditions

# Exercise 4: Miners (cont'd)



- stability conditions for the pairs of player 1 + player $i$:

add up inequalities

$$\left.\begin{aligned} a_1 + a_2 &\geq 1 \\ a_1 + a_3 &\geq 1 \\ &\vdots \\ a_1 + a_n &\geq 1 \end{aligned}\right\} \; n-1 \text{ inequalities}$$

$$(n-1)a_1 + a_2 + \ldots + a_n \geq n-1$$

$$(n-2)a_1 + \sum_{n=1}^{n} a_i \geq n-1$$

$\sum a_i = v(Q)$

$$(n-2)a_1 + v(Q) \geq n-1$$

$v(Q) = n/2$

$$a_1 \geq \frac{n-1-v(Q)}{n-2} = \frac{n-1-\frac{n}{2}}{n-2} = \frac{1}{2}$$

→ the same holds for players 2,3,…,$n$ → all players get ½

# Exercise 4: Miners (cont'd)



- $n$ is odd: the core is empty (!)
  - using the same approach as for even number of players, we obtain:

$$(n-2)\,a_i + v(Q) \geq n-1$$

  - now $v(Q) = (n-1)/2$, which yields

$$a_i \geq \frac{1}{n-2} \cdot \frac{n-1}{2}$$

  - if all players receive this amount, the total is

$$a_1 + a_2 + \ldots a_n = \frac{n}{n-2} \cdot \frac{n-1}{2} = \frac{n}{n-2} \cdot v(Q)$$

  - an imputation that satisfies both $a_1 + a_2 + \ldots a_n = v(Q)$ and the stability conditions doesn't exist

# Exercise 5: Shoes



- for the moment ignore shoe sizes: a pair consists of a left and a right shoe, which can then be sold for €10 (a single shoe on itself is worthless)
- consider a coalition game with 2001 players:
  - 1000 of them have 1 left shoe
  - 1001 of them have 1 right shoe

1. Would you prefer to be a right- or left shoe owner in this game?
   - what ratio of profit shares would you expect for a pair of right- and left shoe owners who combine their shoes and sell the pair?
2. Find the characteristic function for a coalition of $n$ left-shoe owners and $m$ right-shoe owners
3. Find the core of the game
   - Is the core empty? If not, are there multiple imputations in the core, or is there just one?

# Exercise 5: Shoes (cont'd)



- characteristic function for a coalition of $n$ left-shoe owners and $m$ right-shoe owners:

$$v(K) = 10 \times \min(m,n)$$

- grand coalition: $v(Q) = 10{,}000$
- stability conditions for pairs: each pair of a right- and left shoe owner must obtain at least €10 (no requirements for other kinds of pairs)
- this is only possible when all left shoe owners get €10, right shoe owners get nothing (**!**)

→ criticism of the *core* concept

# LECTURE 7:
# COLLUSIVE OLIGOPOLY (CONT'D), COALITION GAMES

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