LECTURE 5:

AUCTIONS

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Games and Decisions

Auctions – A Brief History

- **auctions**: an alternative to take-it-or-leave-it pricing, competition of potential buyers

Babylonian empire (500 BC): auctions of women for marriage

Roman empire: auctions to liquidate the assets of debtors whose property had been confiscated

17th – 18th century, Europe: auctions to sell pieces of art, the birth of many auction houses that still work today:

- *1674*: Stockholm Auction House
- *1744*: Sotheby's
- *1766*: Christie's
- …

Today: online auctions for all kinds of things (*eBay*, *eBid*, *Aukro,…*)

Types of Auctions

Famous types of auctions:

- *English auction* (a.k.a. *open ascending price auction*)
  - the most widespread auction type (the typical art auctions)
  - open bidding, bidders know the others' bids
  - various rule modifications (ending rules – e.g. “auction by candle”)
- *Dutch auction* (a.k.a. *open descending price auction*)
  - the auctioneer cries out gradually descending price bids, the first one to accept the price is the buyer
  - cut flower sales in the Netherlands, perishable goods (fish, tobacco)

- *Envelope auction* (a.k.a. *first-price sealed-bid auctions*)
  - bidders can only submit one bid each (typically, in a sealed envelope)
  - the sale of real estate and securities (used a lot in the post-cummunist countries)

- *Vickrey auction* (a.k.a. *second-price sealed-bid auctions*)
  - “designed” by William Vickrey in 1961
  - used to auction off collectible stamps

Types of Auctions (cont'd)

Basic classification of auction rules:

- *ascending/descending*
  - the direction of bid increments
- *open/sealed-bid*
  - open – bidders submit the bids publicly and after one another
  - sealed-bid – bidders submit the bids secretly and simultaneously
- *first-price/second-price*
  - winner pays the highest/second-highest bid
- *single object/multi-object auction*
  - number of objects auctioned at the same time
- *reserve/no-reserve*
  - the seller can state a reserve price – the minimum price of the auctioned object
  - no-reserve auctions – can attract more bidders (?)
- …

Types of Auctions (cont'd)

Some other auction types:

- *All-Pay Auctions*
  - used for charity auctions
  - various schemes (paying all bids or paying all increments + the whole of the winning bid)
- *Auctions with Buyout Option*
  - the seller can state a buyout price – for immediate purchase
- *Combinatorial Auctions*
  - multi-object auctions, bidding for bundles of objects
  - ferry lines, airport landing slots (it only makes sense to have bundles)
  - winner determination problem, preference expression problems
- *Online Timeshift Auctions*
  - fixed-time English type
  - aim: make bidders bid before the closing timeshift interval

Multi-Object Sealed-Bid Auctions

Basic auction rules:

- bidders submit one bid each
- bids are sealed (= secret) and simultaneous
- first-price auction (winner pays the highest bid)

Additional assumptions (for mathematical modelling):

- *two bidders* only (can be relaxed easily; however, we want to use bi-matrix games as the modelling tool); bidders = *investor 1* and *2*
- investors possess information about the subjective value of each of the n auctioned objects: $s_1, s_2, \dots, s_n$
- total amounts the bidders intend to invest are known: I_1, I_2
- there's a reserve price for each object: $d_1, d_2, \dots, d_n$ (we assume that $s_i \geq d_i$ for $i = 1, 2, \dots, n$)
- in case of equal non-zero bids, the object in question is sold to each of the investors with a probability of ½ (a fair lottery)

Multi-Object Sealed-Bid Auctions (cont'd)

Modelling the auction as a normal-form game:

- *strategy spaces* of the players:

$$X = \left\{ \boldsymbol{x} = (x_1, x_2, \ldots, x_n);\ \sum_{i=1}^{n} x_i = I_1,\ x_i \in \left[d_i, s_i \right] \cup \left\{ 0 \right\} \right\},$$

$$Y = \left\{ \boldsymbol{y} = (y_1, y_2, \ldots, y_n);\ \sum_{i=1}^{n} y_i = I_2,\ y_i \in \left[d_i, s_i \right] \cup \left\{ 0 \right\} \right\}.$$

Multi-Object Sealed-Bid Auctions (cont’d)

Modelling the auction as a normal-form game:

- *strategy spaces* of the players:

$$X = \left\{ \boldsymbol{x} = (x_1, x_2, \ldots, x_n);\ \sum_{i=1}^{n} x_i = I_1,\ x_i \in \left[d_i, s_i\right] \cup \left\{0\right\} \right\},$$

$$Y = \left\{ \boldsymbol{y} = \underbrace{(y_1, y_2, \ldots, y_n)}_{};\ \underbrace{\sum_{i=1}^{n} y_i = I_2,}_{}\ \underbrace{y_i \in \left[d_i, s_i\right] \cup \left\{0\right\}}_{} \right\}.$$

y_i = player 2’s bid for object *i*

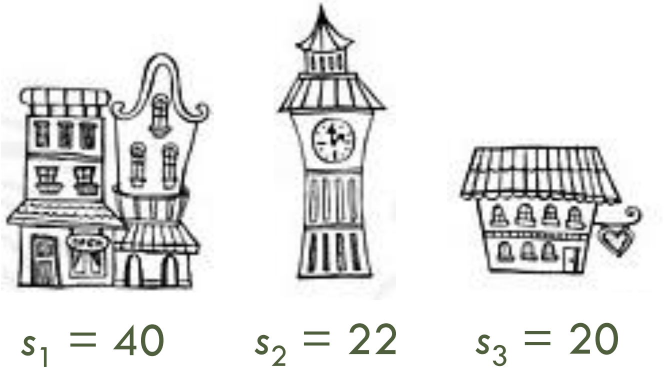
sum of the bids = the intended amount of total investment

it makes sense to bid either 0, or anything between the reserve price and the value of the object

Multi-Object Sealed-Bid Auctions (cont'd)

- in general, these strategy spaces can be *infinite*
- however, it is usually required that the bids must be an integer multiple of a specified monetary unit → *finite* strategy spaces (which enables the bimatrix approach)

Example 1:



- three objects ($n = 3$)
- values: $s_1 = 40$, $s_2 = 22$, $s_3 = 20$
- reserve price 10 for all objects ($d_1 = d_2 = d_3 = 10$)
- total investment: $I_1 = 20$, $I_2 = 10$
- bids must be integer multiples of 10
- strategy spaces (expressed in multiples of 10 for brevity):

$$X = \{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\}$$ ← 6 pure strategies, sum = 2

$$Y = \{(1,0,0),(0,1,0),(0,0,1)\}$$ ← 3 pure strategies, sum = 1

Multi-Object Sealed-Bid Auctions (cont'd)

- *payoff functions*:
  - in order to formulate the payoff functions for both players, we introduce the following functions:

$$\alpha(x,y) = \begin{cases} 1 & \text{for } x > y, \\ \frac{1}{2} & \text{for } x = y, \\ 0 & \text{for } x < y \text{ or } x = y = 0. \end{cases} \qquad \beta(x,y) = \begin{cases} 1 & \text{for } x < y, \\ \frac{1}{2} & \text{for } x = y, \\ 0 & \text{for } x > y \text{ or } x = y = 0. \end{cases}$$

  - note: $\alpha(x_i, y_i)$ is the probability of player 1 obtaining object i
    $\beta(x_i, y_i)$ is the probability of player 2 obtaining object i
  - if player 1 obtains ith object, his total profit rises by $s_i - x_i$
  - payoff functions express the *expected total payoff* for the players:

$$Z_1(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{n} (s_i - x_i)\, \alpha(x_i, y_i), \quad Z_2(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{n} (s_i - y_i)\, \beta(x_i, y_i).$$

Multi-Object Sealed-Bid Auctions (cont'd)

Example 1 (cont'd):

$s_1 = 40$ $s_2 = 22$ $s_3 = 20$

- assume $\boldsymbol{x} = (20,0,0)$ and $\boldsymbol{y} = (10,0,0)$, then:
  - player 1 wins object 1
  - player 2 wins nothing

  $\rightarrow Z_1(\boldsymbol{x},\boldsymbol{y}) = s_1 - x_1 = 40 - 20 = 20$

  $Z_2(\boldsymbol{x},\boldsymbol{y}) = 0$
  - using the formula for Z_1:

  $Z_1(\boldsymbol{x},\boldsymbol{y}) = \sum(s_i - x_i)\ \alpha(x_i, y_i) = (40 - \mathbf{20}) \times \mathbf{1} + (22 - \mathbf{0}) \times \mathbf{0} + (20 - \mathbf{0}) \times \mathbf{0}$

  (legend: values, **bids**, **probabilities**)

- in case $\boldsymbol{x} = (10,10,0)$ and $\boldsymbol{y} = (10,0,0)$: player 1 wins object 2; object 1 is decided by a toss of a coin:

$$Z_1(\boldsymbol{x},\boldsymbol{y}) = (40-10)\times\tfrac{1}{2} + (22-10)\times 1 + 0 = 27$$

$$Z_2(\boldsymbol{x},\boldsymbol{y}) = (40-10)\times\tfrac{1}{2} + 0 + 0 = 15$$

Multi-Object Sealed-Bid Auctions (cont'd)

- game-theoretical solution to the bidding problem – NE again:

  A strategy profile ($\boldsymbol{x}$*, $\boldsymbol{y}$*) *with the property that*

$$Z_1(\boldsymbol{x}, \boldsymbol{y}^*) \leq Z_1(\boldsymbol{x}^*, \boldsymbol{y}^*),$$
$$Z_2(\boldsymbol{x}^*, \boldsymbol{y}) \leq Z_2(\boldsymbol{x}^*, \boldsymbol{y}^*)$$

  for all $\boldsymbol{x} \in X$ *and* $\boldsymbol{y} \in Y$ *is a NE.*

- finite strategy spaces → the auction can be modelled as a bimatrix game
- possible outcomes:
  - unique NE in pure strategies
  - multiple NE's (pure and mixed), no domination
  - multiple NE's (pure and mixed), one dominates the others
  - no pure NE's, (mixed NE's only)

Exercise 1: Unique Pure-Strategy NE

1. Formulate the auction from example 1 as a bimatrix game (i.e., find the payoff matrices for both players, and write them down in a single matrix with double entries).
2. Find the NE of the bimatrix game.

$s_1 = 40$

$s_2 = 22$

$s_3 = 20$

Investor 2

Investor 1

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	20 ; 0		
1,1,0	27 ; 15		
1,0,1			
0,2,0			
0,1,1			
0,0,2			

Exercise 1: Unique Pure-Strategy NE

1. Formulate the auction from example 1 as a bimatrix game (i.e., find the payoff matrices for both players, and write them down in a single matrix with double entries).
2. Find the NE of the bimatrix game.

$s_1 = 40$ $s_2 = 22$ $s_3 = 20$

Investor 2

Investor 1

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	20 ; 0	20 ; 12	20 ; 10
1,1,0	27 ; 15	36 ; 6	42 ; 10
1,0,1	25 ; 15	40 ; 12	35 ; 5
0,2,0	2 ; 30	2 ; 0	2 ; 10
0,1,1	22 ; 30	16 ; 6	17 ; 5
0,0,2	0 ; 30	0 ; 12	0 ; 0

Exercise 2: Multiple NE's – Solvable Case

- consider similar auction as in example 1, only that the values of the object are: $s_1 = 36$, $s_2 = 24$, $s_3 = 20$
- the payoff matrices are in the following table; find *all* NE's for this auction

Investor 2

Investor 1

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	16 ; 0	16 ; 14	16 ; 10
1,1,0	27 ; 13	33 ; 7	40 ; 10
1,0,1	23 ; 13	36 ; 14	31 ; 5
0,2,0	4 ; 26	4 ; 0	4 ; 10
0,1,1	24 ; 26	17 ; 7	19 ; 5
0,0,2	0 ; 26	0 ; 14	2 ; 0

dominated NE

dominating NE

Exercise 3: Multiple NE's

- consider similar auction as in example 1, only that the values of the object are: $s_1 = 26$, $s_2 = 24$, $s_3 = 22$
- find the payoff matrices and *all* NE's for this auction


$s_1 = 26$


$s_2 = 24$


$s_3 = 22$

Investor 2

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	6 ; 0	6 ; 14	6 ; 12
1,1,0	22 ; 8	23 ; 7	30 ; 12
1,0,1	20 ; 8	28 ; 14	22 ; 6
0,2,0	4 ; 16	4 ; 0	4 ; 12
0,1,1	26 ; 16	19 ; 7	20; 6
0,0,2	2 ; 16	2 ; 14	2 ; 0

Investor 1

Exercise 4: No Pure-Strategy NE

- consider similar auction as in example 1, only that the values of the object are: $s_1 = 80$, $s_2 = 24$, $s_3 = 22$
- check that there are no pure-strategy NE's for this auction

$s_1 = 80$

$s_2 = 24$

$s_3 = 22$

Investor 2

Investor 1

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	60 ; 0	60 ; 14	60 ; 12
1,1,0	49 ; 35	77 ; 7	84 ; 12
1,0,1	47 ; 35	82 ; 14	76 ; 6
0,2,0	4 ; 70	4 ; 0	4 ; 12
0,1,1	26 ; 70	19 ; 7	20; 6
0,0,2	2 ; 70	2 ; 14	2 ; 0

Mixed Strategies in Auctions

- from the Nash Existence Theorem, we know that for every bimatrix game there exists at least one NE in mixed strategies
- finding mixed strategies: procedure based on the Equivalence Theorem:

Equivalence Theorem. *Let* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *be* $m \times n$ *matrices with positive elements. The vectors* p^* *and* q^* *are non-zero solution of the nonlinear programming problem*

$$\text{maximize } M(\boldsymbol{p},\boldsymbol{q}) = \boldsymbol{p}^\top(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{q} - \mathbf{1}_m^\top \boldsymbol{p} - \mathbf{1}_n^\top \boldsymbol{q} \tag{1}$$

subject to

$$\begin{aligned} \boldsymbol{A}\boldsymbol{q} &\le \mathbf{1}_m, \\ \boldsymbol{B}^\top \boldsymbol{p} &\le \mathbf{1}_n, \\ \boldsymbol{p} &\ge \mathbf{0}, \\ \boldsymbol{q} &\ge \mathbf{0}. \end{aligned} \tag{2}$$

if and only if $\boldsymbol{x}^* = b\boldsymbol{p}^*$ *and* $\boldsymbol{y}^* = a\boldsymbol{q}^*$ *represent a mixed-strategy* NE *of the bimatrix game with matrices* $\boldsymbol{A},\boldsymbol{B}$*, where:*

$$1/b = \mathbf{1}_m^\top \boldsymbol{p}^* = \sum p_i, \quad 1/a = \mathbf{1}_n^\top \boldsymbol{q}^* = \sum q_i, \quad M(\boldsymbol{p}^*,\boldsymbol{q}^*) = 0.$$

Mixed Strategies in Auctions (cont'd)

- although we can solve the model using *MS Excel Solver* again, there are several problems:
  - non-linear optimization problems may have multiple local extremes, it's advisable to run the algorithm from *different starting points*
  - to solve the auction, we need the equilibrium to be *unique* (or dominant)
  - unfortunately, there are no efficient ways of testing the uniqueness of a mixed strategy equilibrium
- a "relatively reliable" procedure of finding a mixed-strategy NE:
  - **Step 1**: solve the optimization problem

    $$\text{maximize } \mathbf{1}_m^\top \boldsymbol{p} + \mathbf{1}_n^\top \boldsymbol{q} = \sum_{i=1}^{m} p_i + \sum_{j=1}^{n} q_j \text{ subject to (2),}$$

    keep the optimal solution from step 1 as the starting point for step 2
  - **Step 2**: solve the problem: maximize M($\boldsymbol{p}$,$\boldsymbol{q}$) subject to (2); denote optimal values of $\boldsymbol{p}$ and $\boldsymbol{q}$ as $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$
  - **Step 3**: normalize $\boldsymbol{p}^*$ and $\boldsymbol{q}^*$ from step 2 in order to get NE mixed strategies $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ (note: normalize a vector = divide by the sum of its elements)

Collusive Auctions

- *collusion* = secret agreement, conspiracy
- *aim of auctions*: generate the maximum revenue for the seller; works only if the bidders compete

→ collusion is usually not accepted by the auction rules

- modelling approach: cooperative bimatrix games with transferable payoffs

Investor 2

Investor 1

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	60 ; 0	60 ; 14	60 ; 12
1,1,0	49 ; 35	77 ; 7	84 ; 12
1,0,1	47 ; 35	82 ; 14	76 ; 6
0,2,0	4 ; 70	4 ; 0	4 ; 12
0,1,1	26 ; 70	19 ; 7	20; 6
0,0,2	2 ; 70	2 ; 14	2 ; 0

Collusive Auctions (cont'd)

- in order to find the core of the game, we first need: $v(1)$, $v(2)$, and v(1,2)
- finding guaranteed payoffs: *eliminate strictly dominated strategies first!*
  - $v(1) = 60$
  - $v(2) = 7$

Investor 2

Investor 1

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	60 ; 0	60 ; 14	60 ; 12
1,1,0	49 ; 35	77 ; 7	84 ; 12
1,0,1	47 ; 35	82 ; 14	76 ; 6
~~0,2,0~~	~~4 ; 70~~	~~4 ; 0~~	~~4 ; 12~~
~~0,1,1~~	~~26 ; 70~~	~~19 ; 7~~	~~20; 6~~
~~0,0,2~~	~~2 ; 70~~	~~2 ; 14~~	~~2 ; 0~~

Collusive Auctions (cont'd)

- maximum total payoff = $v(1,2) = 96$
  (*note: dominated strategies are included here!*)
- core of the game:

$$\begin{aligned} a_1 + a_2 &= 96, \\ a_1 &\geq 60, \\ a_2 &\geq 7. \end{aligned}$$

- superadditive effect: $v(1,2) - v(1) - v(2) = 96 - 60 - 7 = 29$

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	60 ; 0	60 ; 14	60 ; 12
1,1,0	49 ; 35	77 ; 7	84 ; 12
1,0,1	47 ; 35	82 ; 14	76 ; 6
0,2,0	4 ; 70	4 ; 0	4 ; 12
0,1,1	26 ; 70	19 ; 7	20; 6
0,0,2	2 ; 70	2 ; 14	2 ; 0

1 \ 2	1,0,0	0,1,0	0,0,1
2,0,0	60	74	72
1,1,0	84	84	**96**
1,0,1	82	**96**	82
0,2,0	74	4	16
0,1,1	**96**	26	26
0,0,2	72	16	2

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