Lecture 4:

Mixed Strategies (cont'd), Bimatrix Games

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Games and Decisions

Mixed Strategies in Matrix Games (revision)

- *mixed strategy*: the player decides about the probabilities of the alternative strategies (*sum of the probabilities* = 1); when the decisive moment comes, he/she makes a random selection of the strategy with the stated probabilities
- *notation*: mixed strategies = column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, ith element is the probability of ith row/column of matrix $\boldsymbol{A}$ being picked:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

- payoffs become *random variables*; decisions use *expected payoffs*:

$$\mathrm{E}Z_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i a_{ij} y_j = \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y} = -\mathrm{E}Z_2$$

Mixed-Strategy NE (revision)

- **mathematical definition:**

  NE is a combination of (mixed) strategies $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ with the property that

  $$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^* \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$$

  for all mixed strategies $\boldsymbol{x}$ and $\boldsymbol{y}$.

- **value of the game** (v): player 1's expected payoff at NE ($\boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^*$)

- **Basic Theorem on Matrix Games**: *for any matrix $\boldsymbol{A}$ there exists a mixed-strategy NE.*

- finding mixed-strategy NE's:
  - graphical solution ($2 \times n$ and $m \times 2$ matrices only)
  - linear programming (general $m \times n$ case)

Finding NE – Linear Programming (revision)

- **Step 1**: If there is a negative element in the payoff matrix, make all elements of the matrix positive by adding the same positive number to all elements of the matrix. (This *does* changes the game, but only into a *strategically equivalent* one.)

- **Step 2**: Solve the linear programming problem

  maximize $p_1 + p_2 + \ldots + p_n$

  subject to

$$
\begin{aligned}
a_{11} p_1 + a_{12} p_2 + \ldots + a_{1n} p_n &\leq 1,\\
a_{21} p_1 + a_{22} p_2 + \ldots + a_{2n} p_n &\leq 1,\\
&\cdots\cdots\\
a_{m1} p_1 + a_{m2} p_2 + \ldots + a_{mn} p_n &\leq 1,\\
p_i \geq 0, \quad i &= 1,\ldots,n.
\end{aligned}
$$

- **Step 3**: Divide the primal and dual solutions by the optimal value of the objective function:
  - the *primal solution* determines the strategy of *player 2*.
  - the *dual solution* determines the strategy of *player 1*.

Finding NE – Linear Programming (cont'd)

- note: if we use the symbol $\mathbf{1}_n$ to denote vector

$$\mathbf{1}_n = \left.\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right\} n \text{ elements}$$

we can simplify the LP problem from step 2 as

$$\begin{aligned} &\text{maximize } z = \mathbf{1}_n^\top \boldsymbol{p} \\ &\text{subject to} \\ &\qquad \boldsymbol{A}\boldsymbol{p} \le \mathbf{1}_m, \\ &\qquad \boldsymbol{p} \ge \mathbf{0}. \end{aligned}$$

Proof of the Basic Theorem on Matrix Games

- **Step 1**: show that without loss of generality, we can assume that the elements of matrix $\boldsymbol{A}$ are all positive.
- **Step 2**: show that the following conditions for NE existence are equal (i.e., find *simpler, but equal* versions of NE conditions):

(*C1*) There exist $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ such that $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \leq \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^* \leq \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$ for all mixed strategies $\boldsymbol{x}$ and $\boldsymbol{y}$.

$\Updownarrow$

(*C2*) There exist $\boldsymbol{x}^*$, $\boldsymbol{y}^*$ and v such that $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \leq v \leq \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$ for all mixed strategies $\boldsymbol{x}$ and $\boldsymbol{y}$.

$\Updownarrow$

(*C3*) There exist $\boldsymbol{x}^*$, $\boldsymbol{y}^*$ and v such that $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \leq v \leq \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$ for all *pure* strategies $\boldsymbol{x}$ and $\boldsymbol{y}$.

- **Steps 3** and **4**: prove the existence of $\boldsymbol{x}^*$, $\boldsymbol{y}^*$ and v satisfying (*C3*) using the linear programming Duality Theorem.

Proof of the Basic Theorem (cont'd)

Step 1: *WLOG, all elements of the payoff matrix **A** can be assumed to be positive.*

- if there's a negative element, we can turn the game into a *strategically equivalent* one with positive elements by adding a sufficiently large constant c to all elements of $\boldsymbol{A}$ (thus obtaining matrix $\boldsymbol{A}'$)
- *mathematically*: $\boldsymbol{A}' = \boldsymbol{A} + c\boldsymbol{E}$, where $\boldsymbol{E} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$
- it's easy to see that $\boldsymbol{x}^\top \boldsymbol{A}' \boldsymbol{y} = \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y} + c$ (no matter what the strategies are, player 1 gets an extra payoff of c), hence

$$\boldsymbol{x}^\top \boldsymbol{A}' \boldsymbol{y}^* \le \boldsymbol{x}^{*\top} \boldsymbol{A}' \boldsymbol{y}^* \le \boldsymbol{x}^{*\top} \boldsymbol{A}' \boldsymbol{y} \Leftrightarrow \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y}^* + c \le \boldsymbol{x}^{*\top} \boldsymbol{A} \boldsymbol{y}^* + c \le \boldsymbol{x}^{*\top} \boldsymbol{A} \boldsymbol{y} + c,$$

so if we find a mixed-strategy NE for $\boldsymbol{A}'$, it is also a mixed-strategy NE for $\boldsymbol{A}$

Proof of the Basic Theorem (cont'd)

Step 2: *the following conditions of NE existence are equal ($\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ denote mixed strategies of the two players, v is a real number):*

(*C1*) *There exist* $\boldsymbol{x}^*$ *and* $\boldsymbol{y}^*$ *such that* $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^* \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$ *for all mixed strategies* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

(*C2*) *There exist* $\boldsymbol{x}^*$, $\boldsymbol{y}^*$ *and* v *such that* $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \le v \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$ *for all mixed strategies* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

(*C3*) *There exist* $\boldsymbol{x}^*$, $\boldsymbol{y}^*$ *and* v *such that* $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \le v \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$ *for all* pure *strategies* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

- first, we prove $(C1) \Leftrightarrow (C2)$
  - $(C1) \Rightarrow (C2)$: if $(C1)$ holds, then $(C2)$ holds as well with $v = \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^*$
  - $(C1) \Leftarrow (C2)$: because the inequality in $(C2)$ holds for *all* mixed strategies $\boldsymbol{x}$ and $\boldsymbol{y}$, it has to hold for $\boldsymbol{x} = \boldsymbol{x}^*$, $\boldsymbol{y} = \boldsymbol{y}^*$ as an instance, which yields: $\boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^* \le v \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^* \quad \Rightarrow \quad v = \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^*$
- next, we prove $(C2) \Leftrightarrow (C3)$. Here, $(C2) \Rightarrow (C3)$ is obvious, as pure strategies are only a special case of mixed strategies.

Proof of the Basic Theorem (cont'd)

- the most difficult part in step 2 is proving $(C2) \Leftarrow (C3)$
- the argument goes as follows:

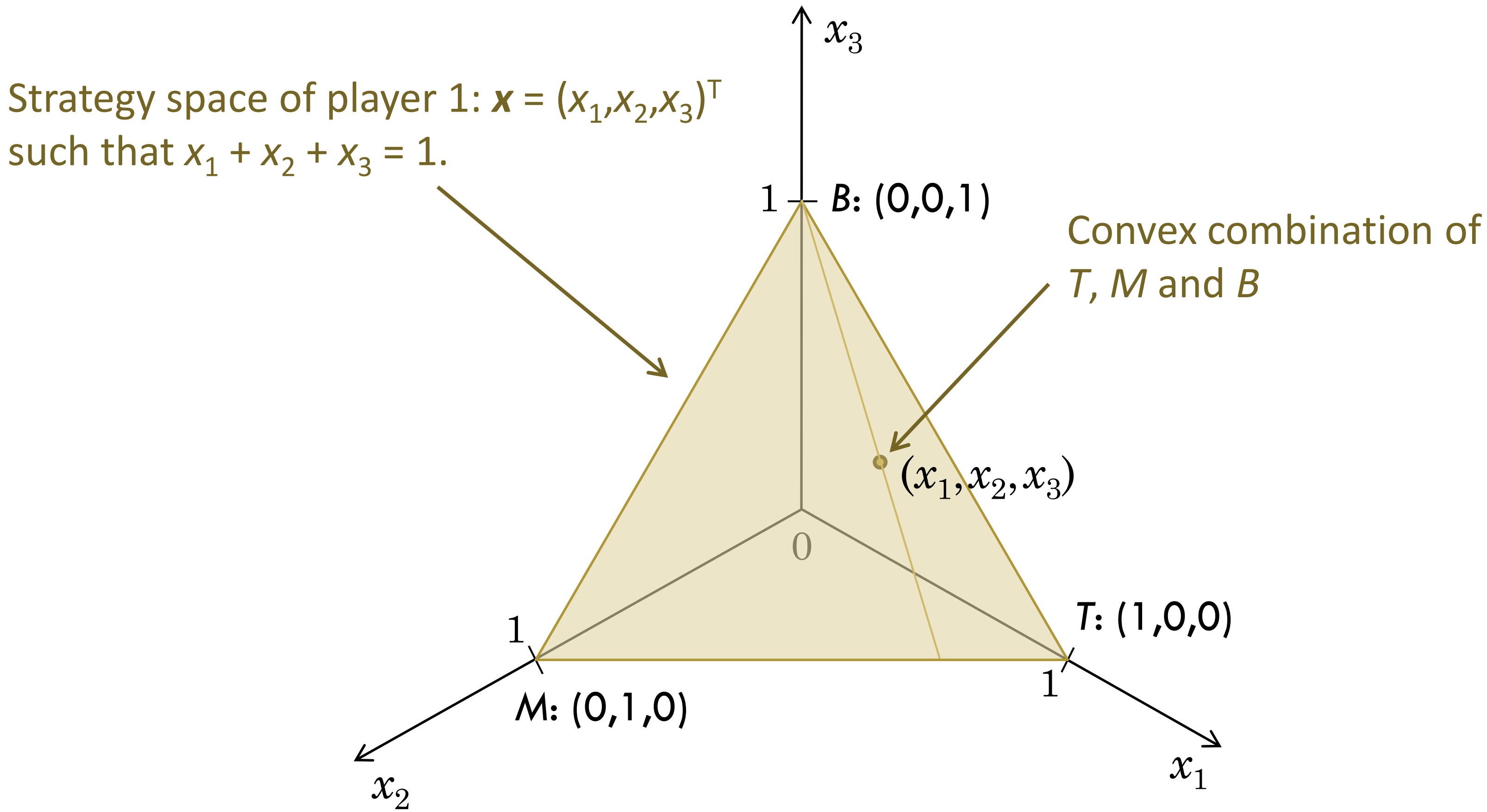
  Mixed strategies are convex combinations of pure strategies; therefore, the expected payoff for a mixed strategy is a convex combination of the expected payoff for pure strategies. Thus, if $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \leq v$ *holds for all pure strategies* $\boldsymbol{x}$*, it has to hold for all mixed strategies as well.*

> Convex combination of *n* vectors:
>
> Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be n vectors of equal size. A *convex combination* of these vectors is a vector $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_n$, where $c_1, c_2, \ldots, c_n$ are real numbers between 0 and 1, the sum of which is 1.

- imagine player 1 has 3 alternative actions (*Top*, *Middle*, and *Bottom* row) → mixed strategies are in the form $\boldsymbol{x} = (x_1, x_2, x_3)^\top$, which can be expressed as a convex combination of pure strategies: $\boldsymbol{x} = x_1 \cdot (1,0,0)^\top + x_2 \cdot (0,1,0)^\top + x_3 \cdot (0,0,1)^\top$

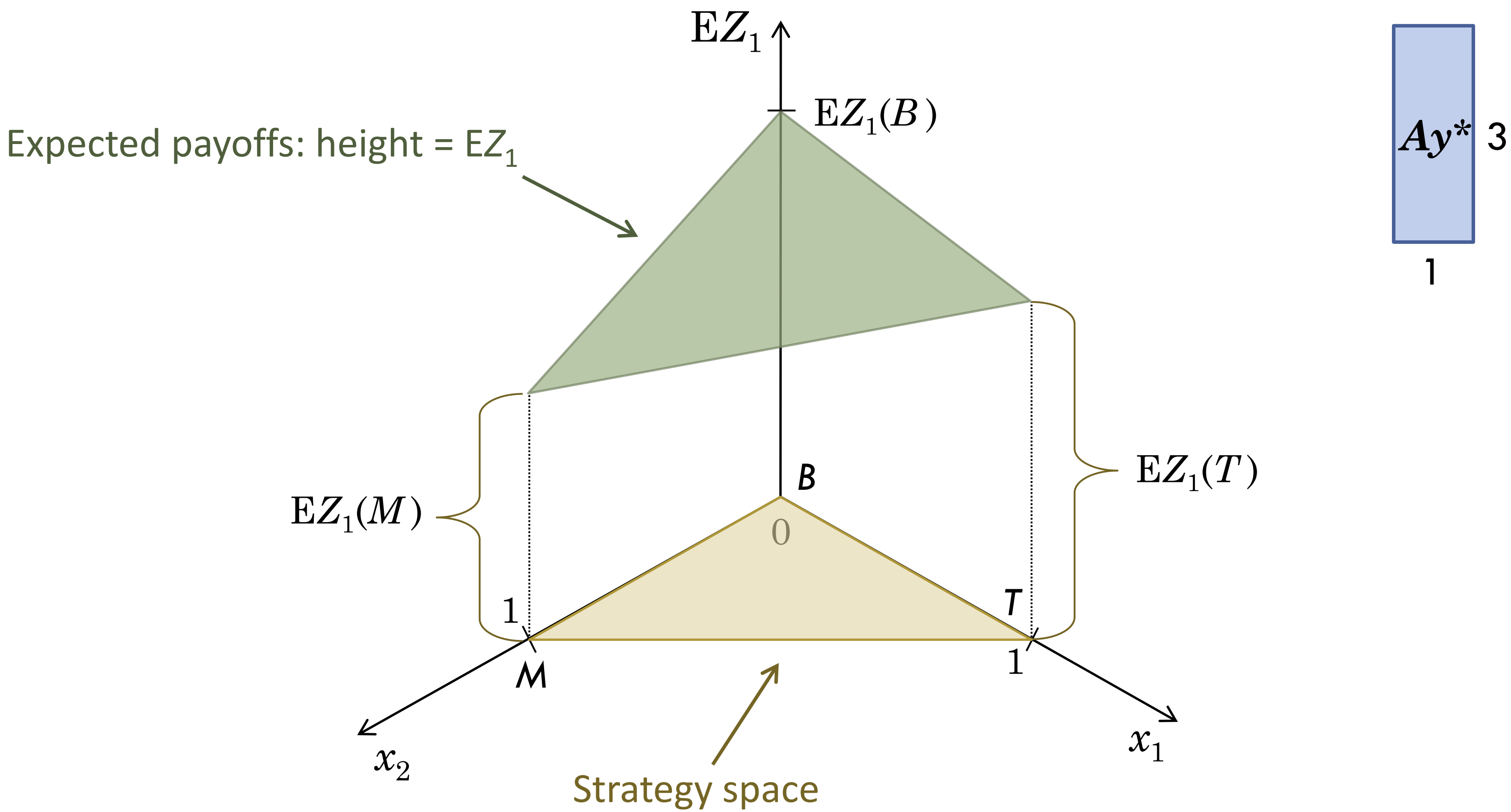
Proof of the Basic Theorem (cont'd)

Graphical illustration of a mixed-strategy space of player 1:



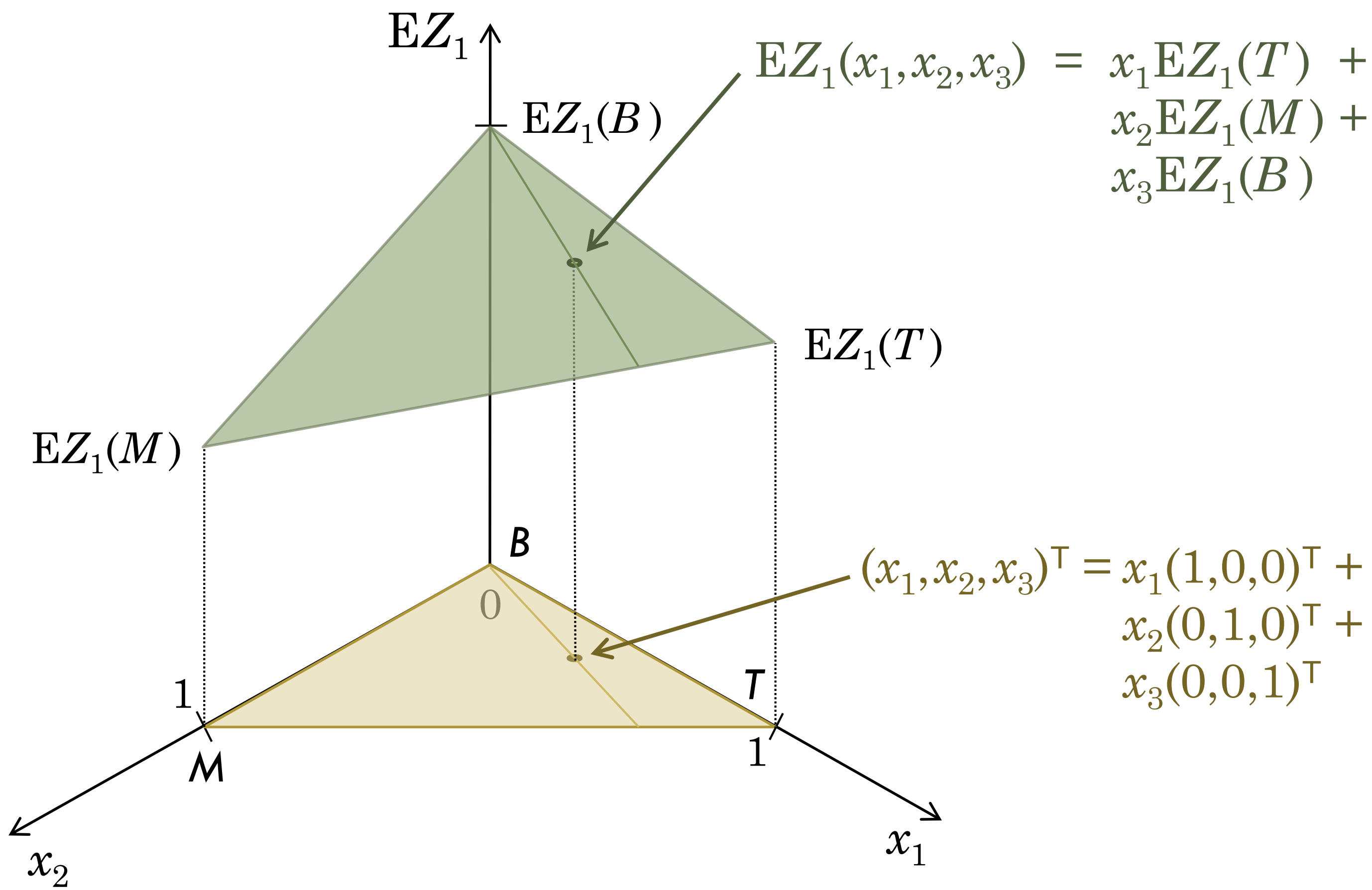
Proof of the Basic Theorem (cont'd)

- including expected payoff in the plot: $x_3 = 1 - x_1 - x_2$, so x_3 needn't be plotted; we plot expected payoff instead: $\mathrm{E}Z_1 = (x_1, x_2, x_3)\boldsymbol{Ay}^*$ $\rightarrow$ linear function of (x_1, x_2, x_3)



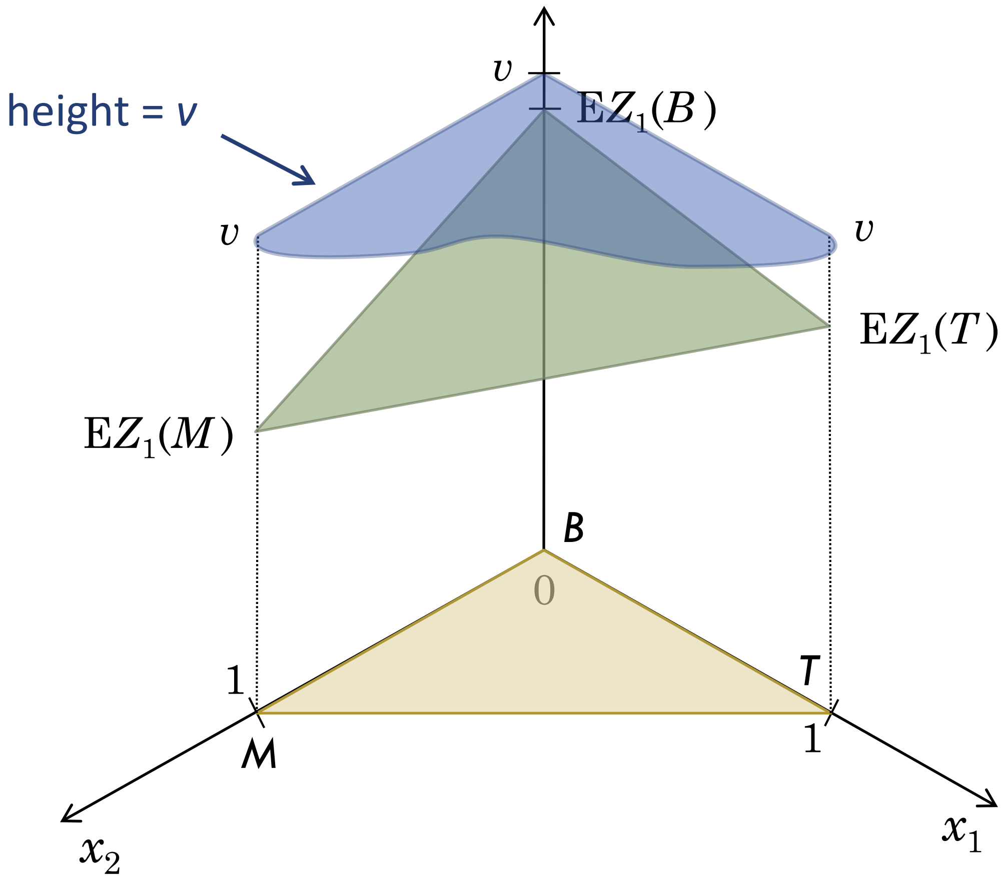
Proof of the Basic Theorem (cont'd)

- the expected payoff for a mixed strategy $(x_1,x_2,x_3)^\mathsf{T}$ is a convex combination of the expected payoff for the pure strategies T, M, and B:



Proof of the Basic Theorem (cont'd)

- therefore, in order to show $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \leq v$ (the height of the whole of the upper triangle is below the level v), it's enough to show it for the three pure strategies (vertices of the triangle)



Proof of the Basic Theorem (cont'd)

- trying to find NE strategy for player 2 → we're looking for $\boldsymbol{y}^* = (y_1,\ldots,y_n)^\top$ such that $\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y}^* \le v$ for all mixed strategies $\boldsymbol{x}$
- from the previous discussion, it suffices for the inequality to hold for all *pure strategies* $\boldsymbol{x}$

Proof of the Basic Theorem (cont'd)

- trying to find NE strategy for player 2 → we're looking for $\boldsymbol{y}^* = (y_1,\ldots,y_n)^\mathsf{T}$ such that $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \le v$ for all mixed strategies $\boldsymbol{x}$
- from the previous discussion, it suffices for the inequality to hold for all *pure strategies* $\boldsymbol{x}$
- *algebraically*: we're looking for $\boldsymbol{y}^* = (y_1,\ldots,y_n)^\mathsf{T}$ such that

$$\begin{array}{c} (1,0,0,\ldots,0)\boldsymbol{A}\boldsymbol{y}^* \le v \\ (0,1,0,\ldots,0)\boldsymbol{A}\boldsymbol{y}^* \le v \\ \vdots \\ (0,0,0,\ldots,1)\boldsymbol{A}\boldsymbol{y}^* \le v \end{array} \Leftrightarrow \begin{array}{c} a_{11}y_1 + a_{12}y_2 + \ldots a_{1n}y_n \le v, \\ a_{21}y_1 + a_{22}y_2 + \ldots a_{2n}y_n \le v, \\ \vdots \\ a_{m1}y_1 + a_{m2}y_2 + \ldots a_{mn}y_n \le v \end{array} \Leftrightarrow \boldsymbol{A}\boldsymbol{y}^* \le \begin{bmatrix} v \\ v \\ \vdots \\ v \end{bmatrix} = v \cdot \mathbf{1}_m$$

and, of course, $\boldsymbol{y}^*$ is a mixed strategy: $y_1 + y_2 + \ldots + y_n = 1$ and $0 \le y_i \le 1$.

Proof of the Basic Theorem (cont'd)

- trying to find NE strategy for player 2 → we're looking for $\boldsymbol{y}^* = (y_1,\ldots,y_n)^\top$ such that $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \le v$ for all mixed strategies $\boldsymbol{x}$
- from the previous discussion, it suffices for the inequality to hold for all *pure strategies* $\boldsymbol{x}$
- *algebraically*: we're looking for $\boldsymbol{y}^* = (y_1,\ldots,y_n)^\top$ such that

$$\begin{array}{c} (1,0,0,\ldots,0)\boldsymbol{A}\boldsymbol{y}^* \le v \\ (0,1,0,\ldots,0)\boldsymbol{A}\boldsymbol{y}^* \le v \\ \vdots \\ (0,0,0,\ldots,1)\boldsymbol{A}\boldsymbol{y}^* \le v \end{array} \quad \Leftrightarrow \quad \begin{array}{c} a_{11}y_1 + a_{12}y_2 + \ldots a_{1n}y_n \le v, \\ a_{21}y_1 + a_{22}y_2 + \ldots a_{2n}y_n \le v, \\ \vdots \\ a_{m1}y_1 + a_{m2}y_2 + \ldots a_{mn}y_n \le v \end{array} \quad \Leftrightarrow \quad \boldsymbol{A}\boldsymbol{y}^* \le \begin{bmatrix} v \\ v \\ \vdots \\ v \end{bmatrix} = v \cdot \mathbf{1}_m$$

  and, of course, $\boldsymbol{y}^*$ is a mixed strategy: $y_1 + y_2 + \ldots + y_n = 1$ and $0 \le y_i \le 1$.
- a similar approach can be used while looking for NE strategy of player 1
  - for player 1, we use the inequality: $v \le \boldsymbol{x}^{*\top}\boldsymbol{A}\boldsymbol{y} = \boldsymbol{y}^\top\boldsymbol{A}^\top\boldsymbol{x}^*$

  → $\boldsymbol{x}$ and $\boldsymbol{y}$ swapped, $\boldsymbol{A}$ transposed, "≥" instead of "≤" (see the next step)

Proof of the Basic Theorem (cont'd)

Step 3: *if a combination of* v, $\boldsymbol{x}^* = (x_1,\dots,x_m)^\mathsf{T}$, *and* $\boldsymbol{y}^* = (y_1,\dots,y_n)^\mathsf{T}$ *satisfies*

$$
\begin{aligned}
a_{11}y_1 + a_{12}y_2 + \dots a_{1n}y_n &\le v, & a_{11}x_1 + a_{21}x_2 + \dots a_{m1}x_m &\ge v,\\
a_{21}y_1 + a_{22}y_2 + \dots a_{2n}y_n &\le v, & a_{12}x_1 + a_{22}x_2 + \dots a_{m2}x_m &\ge v,\\
&\vdots & &\vdots\\
a_{m1}y_1 + a_{m2}y_2 + \dots a_{mn}y_n &\le v, & a_{1n}x_1 + a_{2n}x_2 + \dots a_{mn}x_m &\ge v,\\
y_1, y_2, \dots y_n &\ge 0, & x_1, x_2, \dots x_m &\ge 0,\\
y_1 + y_2 + \dots y_n &= 1, & x_1 + x_2 + \dots x_m &= 1,
\end{aligned}
$$

or, in brief,

$$
\begin{aligned}
\boldsymbol{A}\boldsymbol{y}^* &\le v \cdot \mathbf{1}_m, & \boldsymbol{A}^\top \boldsymbol{x}^* &\ge v \cdot \mathbf{1}_n,\\
\boldsymbol{y}^* &\ge \mathbf{0}, & \boldsymbol{x}^* &\ge \mathbf{0},\\
\mathbf{1}_n^\top \boldsymbol{y}^* &= 1, & \mathbf{1}_m^\top \boldsymbol{x}^* &= 1,
\end{aligned}
$$

then $\boldsymbol{x}^*$ *and* $\boldsymbol{y}^*$ *are the* NE *strategies.*

Proof of the Basic Theorem (cont'd)

Step 4: *there exist* $\boldsymbol{x}^*$ *and* $\boldsymbol{y}^*$ *that satisfy the conditions from step 3 (and, therefore, are the NE strategies).*

- this is the crucial part of the proof; it uses the linear programming Duality Theorem

Primal and Dual LP problems:

Primal problem : maximize $z = \boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},$$
$$\boldsymbol{x} \geq \boldsymbol{0}.$$

Dual problem: minimize $f = \boldsymbol{b}^\top \boldsymbol{y}$
subject to
$$\boldsymbol{A}^\top \boldsymbol{y} \geq \boldsymbol{c},$$
$$\boldsymbol{y} \geq \boldsymbol{0}.$$

Duality Theorem:

If both the primal and the dual problem have *feasible solutions* (i.e., solutions that satisfy the constraints), both have *optimal solutions* as well, and the optimal objective values are equal ($f^* = z^*$)

Proof of the Basic Theorem (cont'd)

- we gradually turn the conditions from step 4 into a primal and dual LP problem; first, divide all the inequalities and equations by v and substitute $p_i = y_i / v$ and $q_j = x_j / v$:

$$\begin{aligned} a_{11}p_1 + a_{12}p_2 + \dots a_{1n}p_n &\le 1, \\ a_{21}p_1 + a_{22}p_2 + \dots a_{2n}p_n &\le 1, \\ &\vdots \\ a_{m1}p_1 + a_{m2}p_2 + \dots a_{mn}p_n &\le 1 \\ p_1, p_2, \dots p_n &\ge 0, \\ p_1 + p_2 + \dots p_n &= 1/v, \end{aligned} \qquad \begin{aligned} a_{11}q_1 + a_{21}q_2 + \dots a_{m1}q_m &\ge 1, \\ a_{12}q_1 + a_{22}q_2 + \dots a_{m2}q_m &\ge 1, \\ &\vdots \\ a_{1n}q_1 + a_{2n}q_2 + \dots a_{mn}q_m &\ge 1, \\ q_1, q_2, \dots q_m &\ge 0, \\ q_1 + q_2 + \dots q_m &= 1/v, \end{aligned}$$

or, in brief,

$$\begin{aligned} \boldsymbol{A}\boldsymbol{p} &\le \boldsymbol{1}_m, \\ \boldsymbol{p} &\ge \boldsymbol{0}, \\ \boldsymbol{1}_n^\top \boldsymbol{p} &= 1/v, \end{aligned} \qquad \begin{aligned} \boldsymbol{A}^\top\boldsymbol{q} &\ge \boldsymbol{1}_n, \\ \boldsymbol{q} &\ge \boldsymbol{0}, \\ \boldsymbol{1}_m^\top \boldsymbol{q} &= 1/v. \end{aligned}$$

Proof of the Basic Theorem (cont'd)

- now we split the conditions for $\boldsymbol{p}$ and $\boldsymbol{q}$ into two linear programming problems, taking the LHS of the last row as the objectives:

maximize $z = \mathbf{1}_n^\top \boldsymbol{p}$
subject to

$$\boldsymbol{A}\boldsymbol{p} \le \mathbf{1}_m, \quad \boldsymbol{p} \ge \mathbf{0}.$$

minimize $f = \mathbf{1}_m^\top \boldsymbol{q}$
subject to

$$\boldsymbol{A}^\top \boldsymbol{q} \ge \mathbf{1}_n, \quad \boldsymbol{q} \ge \mathbf{0}.$$

Proof of the Basic Theorem (cont'd)

- now we split the conditions for $\boldsymbol{p}$ and $\boldsymbol{q}$ into two linear programming problems, taking the LHS of the last row as the objectives:

$$\begin{aligned} &\text{maximize } z = \mathbf{1}_n^\top \boldsymbol{p} \\ &\text{subject to} \\ &\quad \boldsymbol{A}\boldsymbol{p} \le \mathbf{1}_m, \\ &\quad \boldsymbol{p} \ge \mathbf{0}. \end{aligned} \qquad \begin{aligned} &\text{minimize } f = \mathbf{1}_m^\top \boldsymbol{q} \\ &\text{subject to} \\ &\quad \boldsymbol{A}^\top \boldsymbol{q} \ge \mathbf{1}_n, \\ &\quad \boldsymbol{q} \ge \mathbf{0}. \end{aligned}$$

- the two LP problems are in the primal-dual relationship (with $\boldsymbol{b} = \mathbf{1}_m$ an $\boldsymbol{c} = \mathbf{1}_n$)
- both have feasible solutions (take $\boldsymbol{p} = \mathbf{0}$ and $\boldsymbol{q}$ with sufficiently large elements)

Proof of the Basic Theorem (cont'd)

- now we split the conditions for $\boldsymbol{p}$ and $\boldsymbol{q}$ into two linear programming problems, taking the LHS of the last row as the objectives:

maximize $z = \mathbf{1}_n^\top \boldsymbol{p}$
subject to

$$\boldsymbol{A}\boldsymbol{p} \le \mathbf{1}_m,$$
$$\boldsymbol{p} \ge \mathbf{0}.$$

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subject to

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$$\boldsymbol{q} \ge \mathbf{0}.$$

- the two LP problems are in the primal-dual relationship (with $\boldsymbol{b} = \mathbf{1}_m$ an $\boldsymbol{c} = \mathbf{1}_n$)
- both have feasible solutions (take $\boldsymbol{p} = \mathbf{0}$ and $\boldsymbol{q}$ with sufficiently large elements)
- therefore, according to the Duality Theorem, both have optimal solutions ($\boldsymbol{p}^*$ and $\boldsymbol{q}^*$) with equal objective values ($f^* = z^* = 1 / v$).
- it's easy to check that then $\boldsymbol{x}^* = v \cdot \boldsymbol{q}^*$ and $\boldsymbol{y}^* = v \cdot \boldsymbol{p}^*$ satisfy the conditions from step 4, and thus are the NE strategies

Proof of the Basic Theorem (cont'd)

Step 1: *WLOG, all elements of the payoff matrix* $\boldsymbol{A}$ *can be assumed to be positive.*

Step 2: *the following conditions of NE existence are equal :*

(*C1*) $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^* \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$ *for all mixed strategies* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

(*C2*) $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \le v \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$ *for all mixed strategies* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

(*C3*) $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \le v \le \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}$ *for all* pure *strategies* $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

Step 3: *if a combination of* v, $\boldsymbol{x} = (x_1,\ldots,x_m)^\top$, *and* $\boldsymbol{y} = (y_1,\ldots,y_n)^\top$ *satisfies*

$$\begin{aligned} \boldsymbol{A}\boldsymbol{y} &\le v \cdot \boldsymbol{1}_m, & \boldsymbol{A}^\top \boldsymbol{x} &\ge v \cdot \boldsymbol{1}_n, \\ \boldsymbol{y} &\ge \boldsymbol{0}, & \boldsymbol{x} &\ge \boldsymbol{0}, \\ \boldsymbol{1}_n^\top \boldsymbol{y} &= 1, & \boldsymbol{1}_m^\top \boldsymbol{x} &= 1, \end{aligned}$$

then $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are the NE strategies.*

Step 4: *there exist* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *that satisfy the conditions from step 4 (and, therefore, are the NE strategies).*

Finding NE – Linear Programming (revision)

- **Step 1**: If there is a negative element in the payoff matrix, make all elements of the matrix positive by adding the same positive number to all elements of the matrix. (This *does* changes the game, but only into a *strategically equivalent* one.)

- **Step 2**: Solve the linear programming problem

  maximize $p_1 + p_2 + \ldots + p_n$

  subject to

$$
\begin{aligned}
&a_{11}p_1 + a_{12}p_2 + \ldots + a_{1n}p_n \leq 1,\\
&a_{21}p_1 + a_{22}p_2 + \ldots + a_{2n}p_n \leq 1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&a_{m1}p_1 + a_{m2}p_2 + \ldots + a_{mn}p_n \leq 1,\\
&\qquad\qquad p_i \geq 0, \quad i = 1,\ldots,n.
\end{aligned}
$$

- **Step 3**: Divide the primal and dual solutions by the optimal value of the objective function:
  - the *primal solution* determines the strategy of *player 2*.
  - the *dual solution* determines the strategy of *player 1*.

Bimatrix Games

= non-constant-sum games in normal form:

- a finite set of agents: $\{1,2\}$
- strategy spaces (*finite*): $\{X, Y\}$
  - strategy profile: (x,y)
- payoff functions: $Z_1(x,y)$, $Z_2(x,y)$

- payoffs written in two matrices, typically denoted by $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$
  - a_{ij} = *the payoff of player* 1 for strategy profile (i,j) (i.e., player 1 picks i^{th} row and player 2 picks j^{th} column)
  - b_{ij} = *the payoff of player* 2 for strategy profile (i,j)
- typically, $\boldsymbol{A}$ and $\boldsymbol{B}$ written down in a single matrix with double entries:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \quad \boldsymbol{A};\boldsymbol{B} = \begin{bmatrix} 1;5 & 2;6 \\ 3;7 & 4;8 \end{bmatrix}.$$

Prisoner's Dilemma (again…)

- PD is a bimatrix game with matrices

$$\boldsymbol{A} = \begin{bmatrix} -1 & -10 \\ 0 & -5 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -1 & 0 \\ -10 & -5 \end{bmatrix}.$$

Player 2

Player 1

1 \ 2	Stay silent	Betray
Stay silent	-1 ; -1	-10 ; 0
Betray	0 ; -10	-5 ; -5

NE's in Non-Constant-Sum Games

- the same Nash-Equilibrium concept as in case of matrix games (one can't be better off when he/she alone deviates from NE)
- **mathematical definition** (for pure strategies):

A strategy profile (x^*, y^*) *with the property that*

$$Z_1(x, y^*) \leq Z_1(x^*, y^*),$$
$$Z_2(x^*, y) \leq Z_2(x^*, y^*)$$

for all $x \in X$ *and* $y \in Y$ *is a NE.*

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  A strategy profile (x^*, y^*) *with the property that*

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$$Z_2(x^*, y) \le Z_2(x^*, y^*)$$

  for all $x \in X$ *and* $y \in Y$ *is a NE.*

- finding a NE using the *best-response approach*:
  - player 1 plays her best response to the *column* selected by player 2

  → NE has to be the maximum in the column in matrix ***A***
  - player 2 plays her best response to the *row* selected by player 1

  → NE has to be the maximum in the row in matrix ***B***

Prisoner's Dilemma (yet again…)

- player 1's best response:
  - if player 2 stays silent, player 1's best response is to betray. Circle (B,S).
  - if player 2 betrays, player 1's best response is to betray as well. Circle (B,B).

Player 2

Player 1

1 \ 2	Stay silent	Betray
Stay silent	-1 ; -1	-10 ; 0
Betray	0 ; -10	-5 ; -5

Prisoner's Dilemma (yet again...)

- player 2's best response:
  - if player 1 stays silent, player 2's best response is to betray. Square (S,B).
  - if player 1 betrays, player 2's best response is to betray as well. Square (B,B).

Player 2

Player 1

1 \ 2	Stay silent	Betray
Stay silent	-1 ; -1	-10 ; 0
Betray	0 ; -10	-5 ; -5

Prisoner's Dilemma (yet again…)

- (B,B) is the unique NE
- *not* Pareto efficient ((S,S) better for both players)
- for both players, strategy S is strictly dominated by strategy B

Player 2

Player 1

1 \ 2	Stay silent	Betray
Stay silent	-1 ; -1	-10 ; 0
Betray	0 ; -10	-5 ; -5

Mixed-Strategy NE's in Bimatrix Games

- **mathematical definition**:

  NE is a combination of (mixed) strategies $\boldsymbol{x}^*$ *and* $\boldsymbol{y}^*$ *with the property that*

$$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \leq \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^*,$$
$$\boldsymbol{x}^{*\top} \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{x}^{*\top} \boldsymbol{B}\boldsymbol{y}^*$$

  for all mixed strategies $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

Inequalities explained:

$\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y}^* \leq \boldsymbol{x}^{*\top} \boldsymbol{A}\boldsymbol{y}^*$ → can be written as: $EZ_1(\boldsymbol{x},\boldsymbol{y}^*) \leq EZ_1(\boldsymbol{x}^*,\boldsymbol{y}^*)$, which means: If player 1 deviates from NE his/her expected payoff will not increase

$\boldsymbol{x}^{*\top} \boldsymbol{B}\boldsymbol{y} \leq \boldsymbol{x}^{*\top} \boldsymbol{B}\boldsymbol{y}^*$ → can be written as: $EZ_2(\boldsymbol{x}^*,\boldsymbol{y}) \leq EZ_2(\boldsymbol{x}^*,\boldsymbol{y}^*)$, which means: If player 2 deviates from NE his/her expected payoff will not increase

Exercise 1: Battle of the Sexes (again)

- find pure-strategy NE's in the Battle of the Sexes game:

Boy

Girl

Girl \ Boy	Football	Shopping
Football	2 ; 3	0 ; 0
Shopping	1 ; 1	3 ; 2

Exercise 1: Battle of the Sexes (again)

- find pure-strategy NE's in the Battle of the Sexes game:

Boy

Girl

Girl \ Boy	Football	Shopping
Football	2 ; 3	0 ; 0
Shopping	1 ; 1	3 ; 2

- pure-strategy NE's are: (F,F) and (S,S) (*note: different payoffs!*)
- in addition, there's one mixed strategy equilibrium:

$$\boldsymbol{x}^* = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}, \quad \boldsymbol{y}^* = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

Exercise 1: Battle of the Sexes (cont'd)

- assume there's a mixed solution with all elements positive (i.e., $x_1, x_2, y_1, y_2 > 0$)
- if the girl best-responds with a mixed strategy, the boy must make her indifferent between F and S with his mixed strategy (why?)
- therefore: $EZ_1(F,\boldsymbol{y}) = 2 \times y_1 + 0 \times (1-y_1) = 1 \times y_1 + 3 \times (1-y_1) = EZ_1(S,\boldsymbol{y})$, and $y_1 = 3/4$
- similarly, $EZ_2(\boldsymbol{x},F) = 3 \times x_1 + 1 \times (1-x_1) = 0 \times x_1 + 2 \times (1-x_1) = EZ_2(\boldsymbol{x},S)$, and $x_1 = 1/4$

Girl \ Boy	Football	Shopping	
Football	2 ; 3	0 ; 0	x_1
Shopping	1 ; 1	3 ; 2	$1 - x_1$
	y_1	$1 - y_1$	

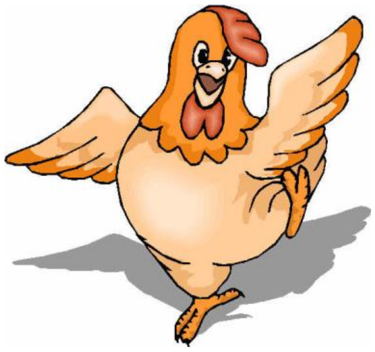
Exercise 2: Game of Chicken

- find NE's in the game of Chicken:
  - two drivers drive towards each other on a collision course
  - either at least one swerves, or both may die in the crash
  - whoever swerves is called "a chicken" (a coward)

Player 2

Player 1

1 \ 2	Swerve	Straight
Swerve	0 ; 0	-1 ; 1
Straight	1 ; -1	-10 ; -10



Exercise 3: Dominated NE's

- find pure-strategy NE's in the following game
- which of the two NE's would you choose if you were player 1?
- which of the two NE's would you choose if you were player 2?

Player 2

Player 1

1 \ 2	L	R
T	7 ; 9	-2 ; -1
B	-2 ; 0	6 ; 4

- the NE (B,R) is dominated by NE $(T,L) \rightarrow (T,L)$ is strategically more credible

Exercise 4: Only Mixed NE's

- find out if there are any pure-strategy NE's in the following bimatrix game
- if not, find a mixed-strategy NE the way we used for the Battle of Sexes

Player 2

Player 1

1 \ 2	L	R
T	3 ; 5	2 ; -1
B	4 ; 2	-2 ; 5

NE Existence in Bimatrix Games

- **Nash Existence Theorem** (John Nash, 1950): *Every normal-form game with finite strategy spaces has a mixed-strategy NE.*

- possible scenarios for bimatrix games:
  - unique NE in pure strategies (*prisoner's dilemma*)
  - multiple NE's (pure and mixed), no domination (*BoS*)
  - multiple NE's (pure and mixed) with domination (*Ex. 3*)
  - no pure NE's, (mixed NE's only) (*Ex. 3*)

- *note: apart from the 2×2 case, mixed NE's are generally difficult to find (non-linear programming techniques)*

Dominated strategies in Bimatrix Games

- as in matrix games, dominated strategies can be eliminated to simplify the problem
- however, it's only safe to eliminate *strictly* dominated strategies (as opposed to only *weakly* dominated ones)

Example 1: *prisoner's dilemma* (yes, indeed, yet again…)

- strategy *Stay silent* is *strictly dominated* for both players
- it doesn't matter whether we start eliminating rows or columns, we always end up with the unique NE:

Player 2 (columns); Player 1 (rows)

1 \ 2	Stay silent	Betray
~~Stay silent~~	~~-1 ; -1~~	~~-10 ; 0~~
Betray	0 ; -10	-5 ; -5

Example 2:

- M weakly dominates T and B
- two different elimination processes:
  - **1** eliminates T, **2** eliminates L → (**2**;**1**)
  - **1** eliminates B, **2** eliminates R → (**1**;**1**)

Player 2

Player 1

1 \ 2	L	R
T	1 ; 1	0 ; 0
M	1 ; 1	2 ; 1
B	0 ; 0	2 ; 1

Cooperation in Bimatrix Games

- so far, we assumed the players do not cooperate
- *note: in* matrix *games, no cooperation is possible (why?)*
- with cooperation, NE is not the relevant principle anymore; still, it can be used in the decision-making process as a certain bargaining tool (or as a benchmark describing the case the players fail to agree on cooperation, see below)
- two different cooperation settings:
  - cooperation with transferable payoffs
  - cooperation with non-transferable payoffs

  → in both cases, players cooperate only of it pays for both; i.e., both earn more than in the non-cooperative setting
    - what is the non-cooperative payoff?
    1. the NE payoff (if this can be decided)
    2. the guaranteed payoff (bully-proof)

Guaranteed Payoff: An Example

Player 2

Player 1

1 \ 2	L	M	R
T	3 ; 1	9 ; -10	9 ; 2
C	-9 ; 9	-5 ; 35	10 ; -2
B	-10 ; 9	13 ; 4	5 ; 4

- the *guaranteed payoff for a strategy* is the worst possible result:
  - for player 1, the worst-case scenarios for the individual strategies are T: **3**, C: –9, B: –10 → *guaranteed payoff of player* 1 = **3**
  - for player 2, we have L: **1**, M: –10, R: –2 → guaranteed payoff = **1**
- guaranteed payoff of player 1/2 is *the largest row/column minimum*
- note that there's no NE in pure strategies here

Guaranteed Payoff: Another Example

Player 2

Player 1

1 \ 2	L	M	R
T	3 ; 1	9 ; -10	9 ; 2
~~C~~	~~-9 ; 9~~	~~-5 ; 35~~	~~2 ; -2~~
B	-10 ; 9	13 ; 4	5 ; 4

- when deciding about the guaranteed payoffs, one can leave out strictly dominated strategies of both players (*implausible bullying*)
- leaving out strategy C of player 1 increases player 2's guaranteed profit to 2
- *notation*: non-cooperative (i.e., NE or guaranteed) profits will be denoted as $v(1)$ for player 1 and $v(2)$ for player 2

Cooperation with Transferable Payoffs

- switch of players focus: from individual payoffs to the total payoff (which can be redistributed afterwards):

1 \ 2	L	M	R
T	3 ; 1	9 ; -10	9 ; 2
C	-9 ; 9	-5 ; 35	2 ; -2
B	-10 ; 9	13 ; 3	5 ; 4

1 \ 2	L	M	R
T	4	-1	11
C	0	**30**	0
B	-1	16	9

- the maximum attainable total payoff $= v(1,2) = 30$
- crucial question: how to divide the total payoff?

Distribution of Payoffs

- **imputation**: a potential final distribution of payoffs to both players (a_1 for player 1, a_2 for player 2)
- **core of the game**: the set of all imputations (a_1,a_2) such that:

$$a_1 + a_2 = v(1,2),$$
$$a_1 \geq v(1),$$
$$a_2 \geq v(2),$$

e.g., for the game from the previous slides,

$$a_1 + a_2 = 30,$$
$$a_1 \geq 3,$$
$$a_2 \geq 2.$$

- **superadditive effect**: $v(1,2) - v(1) - v(2) = 30 - 3 - 2 = 25$
- a fair division: each player gets her guaranteed payoff + half of the superadditive effect:

$$a_1{}^* = 3 + 25/2 = 15.5, \quad a_2{}^* = 2 + 25/2 = 14.5.$$

Lecture 4:
Mixed Strategies (cont'd), Bimatrix Games

Jan Zouhar

Games and Decisions