# LECTURE 3: MIXED STRATEGIES

Jan Zouhar Games and Decisions

#### Matrix Games (revision)

a special case of zero-sum games:	
a finite set of agents:	$\{1,2\}$
strategy spaces (finite):	$\{X,Y\}$
strategy profile:	(x,y)
payoff functions:	$Z_1(x,y), Z_2(x,y)$
□ zero-sum payoffs: $Z_1(x,y) + Z_2(x,y) = 0$	

payoffs written in a matrix, typically denoted by *A*: 

$$\boldsymbol{A} = (a_{ij})_{\substack{i=1,\dots,m\\j=1,\dots,n}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

 $a_{ii} = the payoff of player 1$  for strategy profile (i,j)(i.e., player 1 picks *i*th strategy and player 2 picks *j*th)

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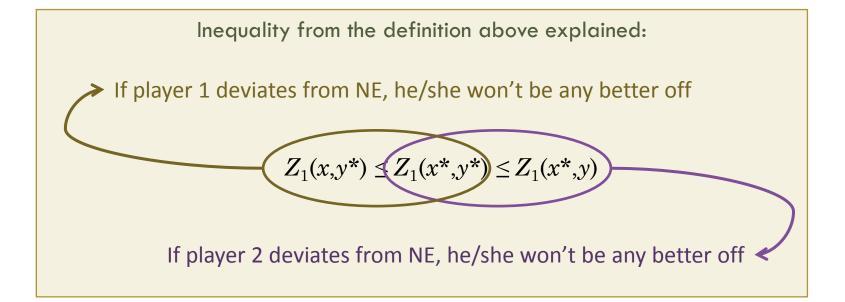
## Nash Equilibrium in Matrix Games (revision)

#### mathematical definition:

A strategy profile  $(x^*, y^*)$  with the property that

$$Z_1(x,y^*) \le Z_1(x^*,y^*) \le Z_1(x^*,y)$$

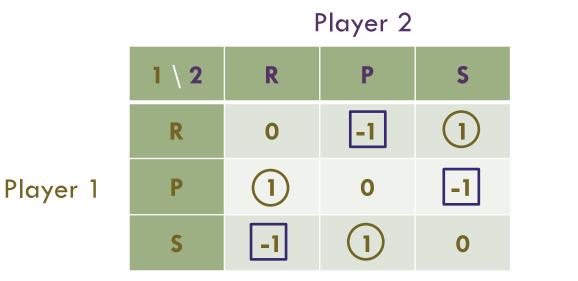
for all  $x \in X$  and  $y \in Y$  is a NE.

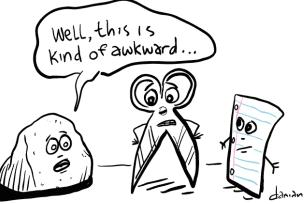


#### Matrix Games with No Pure Strategy NE

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#### Rock, Paper, Scissors



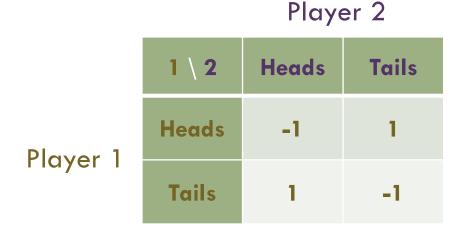


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#### Matrix Games with No Pure Strategy NE (cont'd)

#### Matching pennies game

- both players secretly turn a coin to heads/tails
- □ if both heads or both tails, player 1 pays \$1, otherwise player 2 pays \$1





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#### Matrix Games with No Pure Strategy NE (cont'd)

Penalty kicks (a modified version of matching pennies)

- $\Box$  Kick vs. Goalkeeper
- □ *strategies*: Right/Left
- payoffs: scoring probabilities

	1 \ 2	Right	Left
Kick	Right	0.5	0.9
	Left	0.9	0.5

#### Goalkeeper



#### Matrix Games with No Pure Strategy NE (cont'd)

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- stability in NE: even if he knows his opponent is playing the NE strategy, player 1 has no incentive to deviate from NE (the same goes for player 2)
- matching pennies:
  - if player one knows player 2 is playing Heads, he can win by playing Tails and vice versa
  - $\rightarrow$  neither Heads nor Tails can be the NE strategy
- $\Box$  still, there is a rational way to play the game:
  - each time you play the game, toss the coin and let it decide the strategy on itself
  - even if your opponent knows your strategy, she can't take advantage of that (compare with *RPS* game)
  - tossing a coin means applying a *mixed strategy*

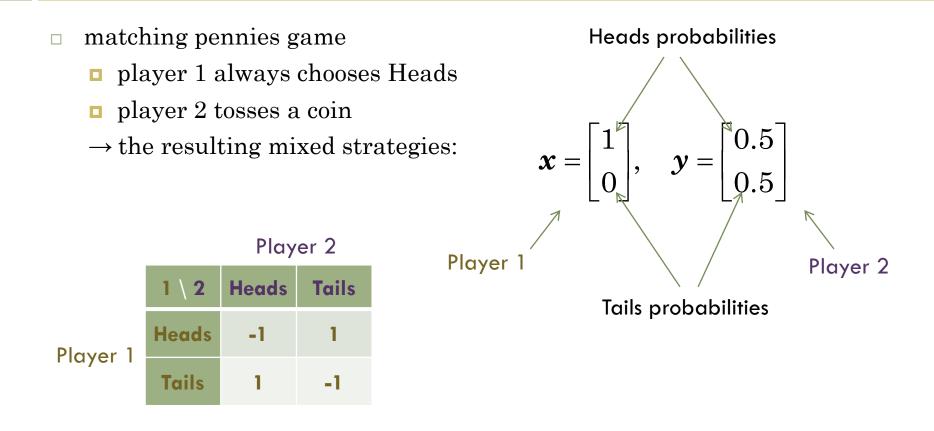
#### Pure Vs. Mixed Strategies in Matrix Games

- $\Box$  switch from *pure strategies* to *mixed strategies*:
  - *pure strategy*: the player decides for a particular strategy (i.e., picks a certain row or column)
  - mixed strategy:
    - the player decides about the probabilities of the alternative strategies
      (aum of the probabilities = 1)

(sum of the probabilities = 1)

- when the decisive moment comes, he/she makes a random selection of the strategy with the stated probabilities
- allowing for mixed strategies in a matrix game: sometimes called a mixed extension of a matrix game
- notation: mixed strategies = column vectors x and y, ith element represents the probability of ith row/column being picked

## Mixed Strategies: An Example



 $\Box$  note: **x** actually is a pure strategy here

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## Payoffs with Mixed Strategies

- when using mixed strategies, the payoffs for the individual players become *random variables*
  - the possible values are stated in the game's matrix
  - the final outcome depends on the strategies eventually picked by the individual players
- in order to be able to treat the payoffs resulting from a combination of mixed strategies as a single number, we use the concept of *expected payoff*

## Payoffs with Mixed Strategies

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#### Expected value:

Let *Z* be a discrete-valued random variable with possible values in  $\Omega$ . The *expected value of Z* is:

$$\mathbf{E} Z = \sum_{z \in \Omega} z \cdot \Pr \left\{ Z = z \right\}$$

## **Expected Payoffs: An Example**

consider matching pennies again, with x = [1 0]<sup>T</sup>, y = [0.5 0.5]<sup>T</sup>
 the expected payoff of player one is:

$$\begin{split} \mathbf{E}Z_{1} &= \begin{cases} -1 \times \Pr(H, H) &+ 1 \times \Pr(H, T) \\ +1 \times \Pr(T, H) &- 1 \times \Pr(T, T) \end{cases} \\ &= \begin{cases} -1 \times (1 \times 0.5) &+ 1 \times (1 \times 0.5) \\ +1 \times (0 \times 0.5) &- 1 \times (0 \times 0.5) \end{cases} \\ &= \begin{cases} -0.5 &+ 0.5 \\ +0 &+ 0 \end{cases} = 0 \end{split}$$

• the expected value of player 2 is:  $EZ_2 = -EZ_1 = 0$ 

### **Expected Payoffs: A Generalization**

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- $\Box \quad \text{consider a general } 2 \times 2 \text{ matrix game with matrix } A \text{ and mixed strategies} \\ x \text{ and } y:$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \qquad \begin{array}{c} & \frac{y_1 & y_2}{x_1} \\ x_1 & a_{11} & a_{12} \\ x_2 & a_{21} & a_{22} \end{array}$$

□ the expected payoff of player 1:

$$EZ_{1} = x_{1}a_{11}y_{1} + x_{1}a_{12}y_{2} + x_{2}a_{21}y_{1} + x_{2}a_{22}y_{2} = \sum_{i}\sum_{j}x_{i}a_{ij}y_{j} = \mathbf{x}^{\top}\mathbf{A}\mathbf{y}$$

□ the last two expressions hold for the general case of  $m \times n$  matrix games

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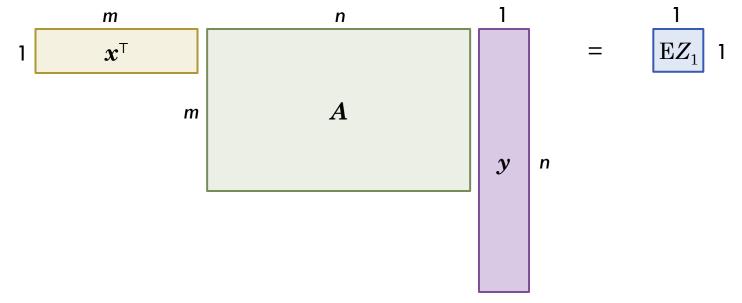
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#### Expected Payoffs: A Generalization (cont'd)

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Expected payoffs in an  $m \times n$  matrix game with mixed strategies:

$$\mathbf{E} Z_1 = \sum_{i=1}^m \sum_{j=1}^n x_i a_{ij} y_j = \mathbf{x}^\top \mathbf{A} \mathbf{y} = -\mathbf{E} Z_2$$



## Expected payoffs: Exercise 1

 calculate the expected payoff for both players in the penalty kicks game for the following mixed strategies of the two players:

**a)** 
$$x = [1 \ 0]^{\mathsf{T}}, \quad y = [0.5 \ 0.5]^{\mathsf{T}}.$$
  
**b)**  $x = [0.5 \ 0.5]^{\mathsf{T}}, \quad y = [0.5 \ 0.5]^{\mathsf{T}}.$ 

**c)**  $x = [0 \ 1]^{\mathsf{T}}, \qquad y = [0.5 \ 0.5]^{\mathsf{T}}.$ 

#### Goalkeeper

	1 \ 2	Right	Left
Kick	Right	0.5	0.9
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#### Nash Equilibrium with Mixed Strategies

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- overall concept: the same as with pure strategies

"NE is such a combination of strategies that neither of the players can increase their payoff by choosing a different strategy."

"NE is a solution with the property that whoever of the players chooses some other strategy, he or she will not increase his or her payoff."

#### mathematical definition:

NE is a combination of (mixed) strategies  $\mathbf{x}^*$  and  $\mathbf{y}^*$  with the property that

$$x^{\top}Ay^* \leq x^{*^{\top}}Ay^* \leq x^{*^{\top}}Ay$$

for all mixed strategies x and y.

□ value of the game: player 1's expected payoff at NE  $(x^* A y^*)$ 

#### Nash Equilibrium with Mixed Strategies (cont'd)

Inequality from the NE definition explained: Can be rewritten as:  $EZ_1(x, y^*) \le EZ_1(x^*, y^*)$ , which means: If player 1 deviates from NE, his/her expected payoff will not increase  $x^{\top}Ay^{*} \leq x^{*^{\top}}Ay^{*} \geq x^{*^{\top}}Ay$ Can be rewritten as:  $-EZ_2(x^*, y^*) \le -EZ_2(x^*, y)$ , or  $EZ_2(x^*, y) \le EZ_2(x^*, y^*)$ , which means: If player 2 deviates from NE, his/her expected payoff will not increase

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#### **Basic Theorem on Matrix Games**

- it is easily seen that a pure-strategy NE is also a mixed-strategy NE; therefore, if a matrix game has a NE in pure strategies, it has a NE in mixed strategies as well
- what happens in the are no pure-strategy NE? As JOHN VON NEUMANN proved in 1928, even if a matrix game has no NE in pure strategies (i.e., no saddle point of the payoff matrix), it still has a NE in mixed strategies (*always*)
- **Basic Theorem on Matrix Games:**

For any matrix A there exist mixed strategies  $x^*$  and  $y^*$  such that

$$x^{\top}Ay^{*} \leq x^{*}Ay^{*} \leq x^{*}Ay$$

for all mixed strategies  $\mathbf{x}$  and  $\mathbf{y}$ .

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## Finding Mixed-Strategy NE's

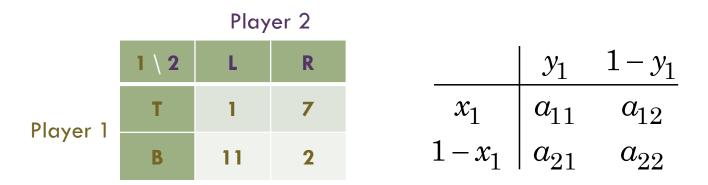
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- Basic Theorem tells us that every matrix game has a mixed-strategy solution
- two possible approaches of finding the solution depending on the size of the matrix A:
  - a) "small"  $-2 \times n$  and  $m \times 2$  matrices:
    - simple
    - graphical solution
  - b) "general"  $m \times n$  matrices:
    - a bit more complicated
    - linear programming
    - can be solved in *MS Excel* (or other SW)

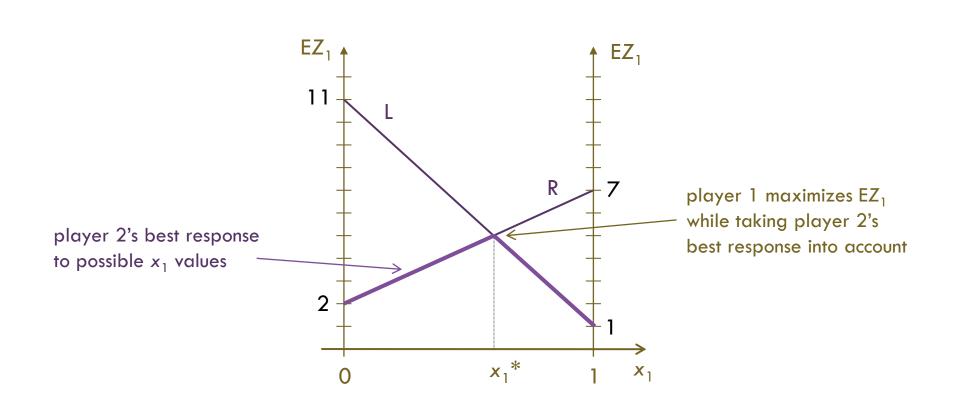
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 $\Box$  2×2 matrix  $\rightarrow$  players actually choose only 1 probability:

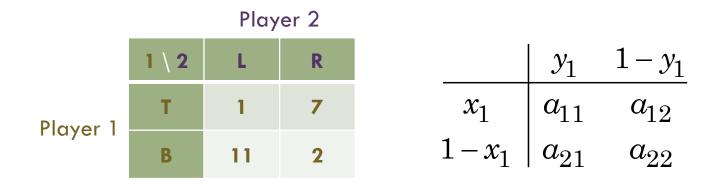
$$x_2 = 1 - x_1, \quad y_2 = 1 - y_1$$

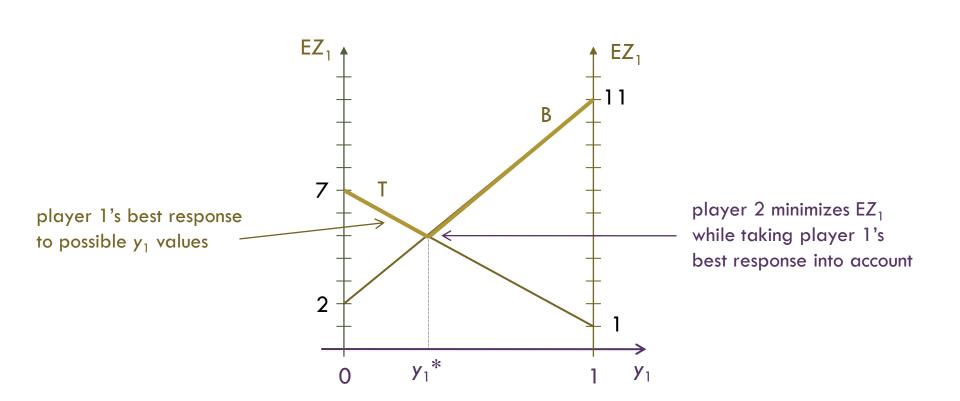
- □ player 1's decision:  $x_1$  (probability of *T*)
  - expected payoffs:
    - player 2 picks *L*:  $EZ_1 = 1 \times x_1 + 11 \times (1 x_1) = 11 10x_1$
    - player 2 picks R:  $EZ_1 = 7 \times x_1 + 2 \times (1 x_1) = 2 + 5x_1$
    - player 2 chooses a mixed strategy: anything in between (more precisely: a *convex combination* of the two)





- □ player 2's decision:  $y_1$  (probability of *L*)
  - expected payoffs:
    - player 1 picks T:  $EZ_1 = 1 \times y_1 + 7 \times (1 y_1) = 7 6y_1$
    - player 1 picks *B*:  $EZ_1 = 11 \times y_1 + 2 \times (1 y_1) = 2 + 9y_1$
    - player 1 chooses a mixed strategy: anything in between (more precisely: a *convex combination* of the two)





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- finding  $x_1^*$  and  $y_1^*$ :
  - graphical solution: intersection of two lines
  - *numerically*: system of two linear equations
    - For player 1:  $EZ_{1} = 11 10x_{1}$   $EZ_{1} = 2 + 5x_{1}$   $EZ_{1} = 5$   $EZ_{1} = 7 6y_{1}$   $EZ_{1} = 2 + 9y_{1}$   $EZ_{1} = 5$
- equilibrium mixed strategies:

$$\boldsymbol{x^*} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}, \quad \boldsymbol{y^*} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

- value of the game:  $x^* A y^* = 5$ 
  - $\hfill\square$  represented by  $\mathbf{E} Z_1$ -value of the intersection in both plots

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### **Graphical Solution: Exercise 2**

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- penalty kicks game, kick better when aiming left
- use graphical solution to find NE strategies for both players



#### Goalkeeper

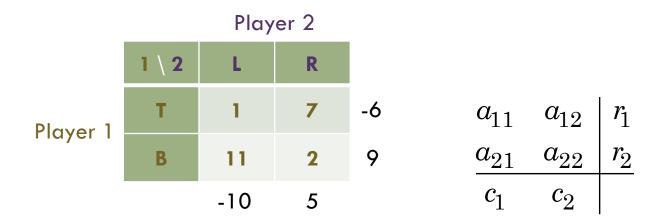
## Row and Column Differences Formula

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- □ the solution of the system of two equations can be expressed easily using *row* and *column differences*:

$$r_1 = a_{11} - a_{12} \quad c_1 = a_{11} - a_{21}$$
$$r_2 = a_{21} - a_{22} \quad c_2 = a_{12} - a_{22}$$

 $\square$  NE strategies:

$$x_1^* = \frac{r_2}{c_2 - c_1}, \quad y_1^* = \frac{c_2}{r_2 - r_1}$$



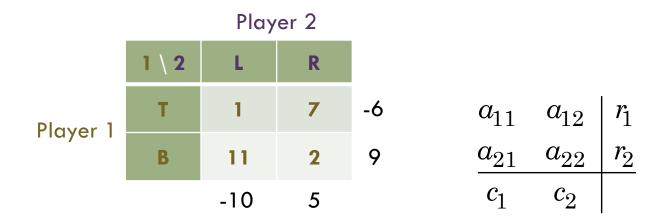
### Row and Column Differences Formula

 derivation of the formula for NE strategies from the systems of equations:

$$L: EZ_{1} = a_{11}x_{1} + a_{21}(1 - x_{1}) = a_{21} + c_{1}x_{1}$$
  

$$R: EZ_{1} = a_{12}x_{1} + a_{22}(1 - x_{1}) = a_{22} + c_{2}x_{1}$$
  

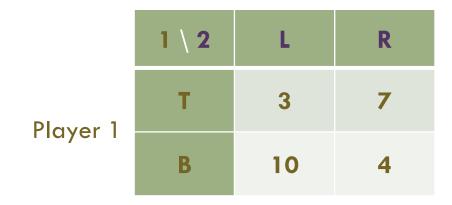
$$x_{1} = \frac{r_{2}}{c_{2} - c_{1}}$$



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## Row and Column Differences: Exercise 3

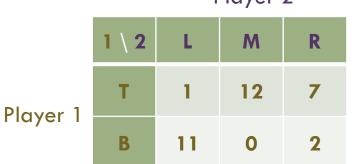
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- using *row* and *column differences* formula to find the NE strategies in the following game





## Graphical Solution – $2 \times n$ Matrices

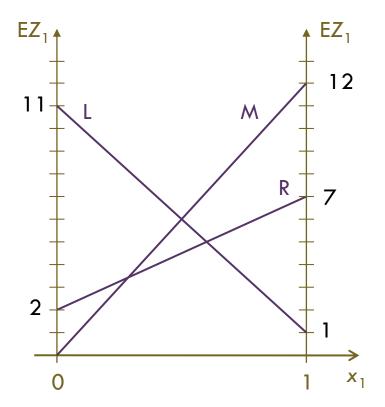
- $\square$  a generalized version of 2×2 graphical solution
  - **start with player 1's plot**
  - determine the "active" best responses (strategies that can be played in a NE)
  - consider only active best responses for player 2's plot
- □ for  $m \times 2$  matrices, proceed similarly



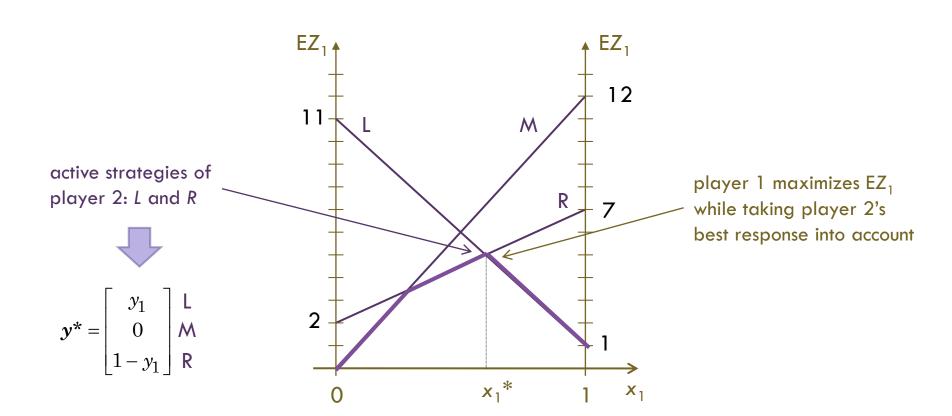


Expected payoffs with $\mathbf{x} = [x_1, 1-x_1]^T$ :			
<i>L</i> : $EZ_1 = 1 \times x_1 + 11 \times (1 - x_1) = 11 - 10x_1$			
<i>M</i> : $EZ_1 = 12 \times x_1 + 0 \times (1 - x_1) = 12x_1$			
<i>R</i> : $EZ_1 = 7 \times x_1 + 2 \times (1 - x_1) = 2 + 5x_1$			

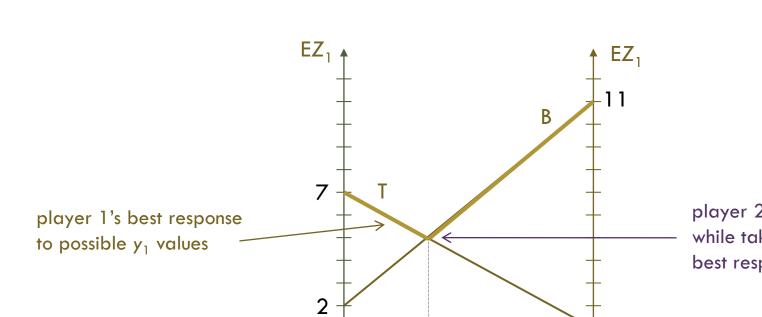
#### Graphical Solution $-2 \times n$ Matrices (cont'd)



#### Graphical Solution $-2 \times n$ Matrices (cont'd)



#### Graphical Solution $-2 \times n$ Matrices (cont'd)



0

**y**<sub>1</sub>\*

player 2 minimizes EZ<sub>1</sub> while taking player 1's best response into account

**y**<sub>1</sub>

## Finding NE – Linear Programming

 the construction of the algorithm and its explanation is actually presented in the proof of the *Basic Theorem on Matrix Games* (see the *Games and Economic Decisions* textbook, or next lecture)

Linear programming:

Using *linear programming* methods, one can finding a maximum or minimum of a linear function of multiple variables on a set given by linear constraints:

## Finding NE – Linear Programming

(cont'd)

- Step 1: If there is a negative element in the payoff matrix, make all elements of the matrix positive by adding the same positive number to all elements of the matrix. (This does changes the game, but only into a *strategically equivalent* one.)
- □ **Step 2**: Solve linear programming problem maximize  $p_1 + p_2 + ... + p_n$ subject to

$$\begin{aligned} a_{11} p_1 + a_{12} p_2 + \dots + a_{1n} p_n &\leq 1, \\ a_{21} p_1 + a_{22} p_2 + \dots + a_{2n} p_n &\leq 1, \\ \dots & \dots & \dots \\ a_{m1} p_1 + a_{m2} p_2 + \dots + a_{mn} p_n &\leq 1, \\ p_i &\geq 0, \quad i = 1, \dots, n. \end{aligned}$$

- Step 3: Divide the primal and dual solutions by the optimal value of the objective function:
  - the *primal solution* determines the strategy of *player 2*.
  - the *dual solution* determines the strategy of *player 1*.

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### Finding NE – Linear Programming

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□ note: if we use the symbol  $\mathbf{1}_n$  to denote vector

$$\mathbf{1}_{n} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\} n \text{ elements}$$

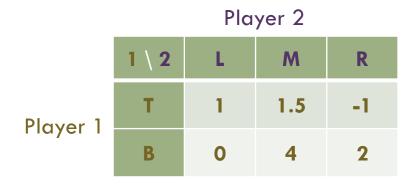
we can simplify the LP problem from step 2 as

maximize 
$$z = \mathbf{1}_n^\top p$$
  
subject to  
 $A p \leq \mathbf{1}_m,$   
 $p \geq \mathbf{0}.$ 

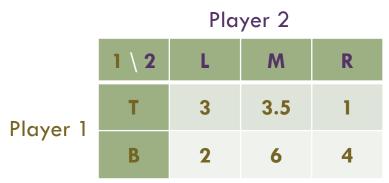
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## Using LP to Find NE: An Example

• we'll find mixed-strategy NE's in the following matrix game:



Step 1: elimination of negative elements. We'll add a constant *c* = 2 to all elements of the matrix (to get a strategically equivalent matrix game with non-negative elements.)



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## Using LP to Find NE: An Example

- Step 2: solve LP problem maximize  $p_1 + p_2 + p_3$ subject to  $3p_1 + 3.5p_2 + 1p_3 \le 1$ ,
  - $\begin{array}{rl} p_1 + b.b \, p_2 + 1 \, p_3 \leq 1, \\ p_1 + & 6 \, p_2 + 4 \, p_3 \leq 1, \\ & p_1, p_2, p_3 \geq 0. \end{array}$
  - solve using MS Excel (to see how, download the NE\_Solver.xls file from my website), optimal values are:
    - objective function value: 0.4
    - primal solution:  $\mathbf{p} = [0.3 \ 0 \ 0.1]^{\mathsf{T}}$
    - dual solution:  $\boldsymbol{q} = [0.2 \ 0.2]^{\mathsf{T}}$
- **Step 3**: the equilibrium strategies are:
  - D player 1:  $x^* = q / 0.4 = [0.5 \ 0.5]^T$
  - **D** player 2:  $y^* = p / 0.4 = [0.25 \ 0 \ 0.75]^T$

(cont'd)

## Using LP to Find NE: An Example

- $\Box$  value of the game is

$$x^{\star \top}Ay^{\star} = 1$$
 / objective function = 1 / 0.4 = 2.5

□ value of the original game is

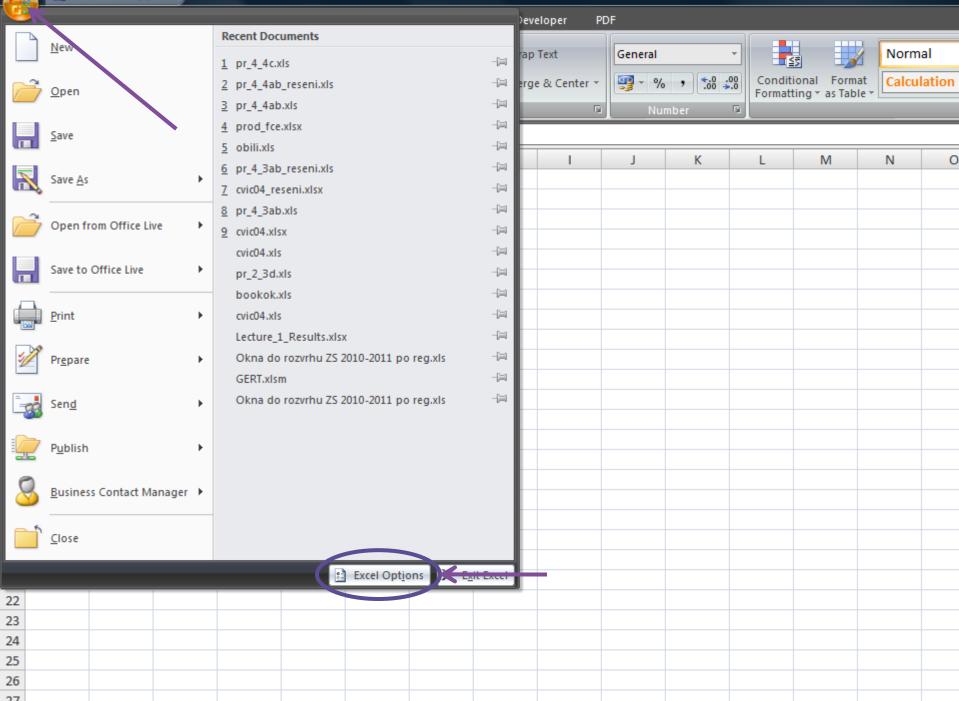
1 / objective function - c = 1 / 0.4 - 2 = 2.5 - 2 = 0.5

following slides: activating the Solver add-in in MS Excel 2007

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# LECTURE 3: MIXED STRATEGIES

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