Lecture 2:
Matrix Games

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Games and Decisions

Normal (Strategic) Form Games

Main traits:

- **simultaneous moves**
  - players have to make their strategy choices *simultaneously, without knowing the strategies* that have been chosen by the other player(s)
- **common knowledge of available strategies**
  - while there is no information about what other players will actually choose, we assume that the *strategic choices* available to each player *are known by all players*
- **rationality & interdependence**
  - players must think not only about their own best strategic choice but also the best strategic choice of the other player(s)

Normal (Strategic) Form Games (cont'd)

- *mathematical notation* of a game's elements:
  - a finite set of agents: $\{1,2,\ldots,n\}$
  - strategy spaces (finite or infinite): $\{X_1, X_2, \ldots, X_n\}$
    - after selection: a strategy profile: $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$
  - payoff functions: $Z_1(\boldsymbol{x}), Z_2(\boldsymbol{x}), \ldots, Z_n(\boldsymbol{x})$

    (or: $Z_1(x_1, x_2, \ldots, x_n), Z_2(x_1, x_2, \ldots, x_n), \ldots, Z_n(x_1, x_2, \ldots, x_n)$)

- *infinite strategy spaces*: payoff functions typically expressed as mathematical formulas:

$$Z_1(x_1,x_2) = 100 - (x_1 + 2x_2),$$

$$Z_2(x_1,x_2) = 150 - (2x_1 + x_2).$$

  - example: *Cournot oligopoly*, strategies = quantities supplied

Normal (Strategic) Form Games (cont'd)

- *finite strategy spaces*: payoffs specified in tables (or *matrices*)
  - Prisoner's dilemma revisited:
    - payoffs in two matrices → a *bimatrix game*

Player B

Player A

A \ B	Stay silent	Betray
Stay silent	–1 , –1	–10 , 0
Betray	0 , – 10	– 5 , – 5

Game 1: A Bimatrix Game

- choose a strategy for player 1 in the following bimatrix game:

Player 2

Player 1

1 \ 2	W	X	Y	Z
A	5 , 2	2 , 6	1 , 4	1 , 4
B	9 , 5	1 , 3	0 , 2	4 , 8
C	7 , 0	2 , 2	1 , 5	5 , 1
D	0 , 0	3 , 2	2 , 1	1 , 1

Game 1: A Bimatrix Game (cont'd)

- comments on individual strategies:
  - **A**: no matter what player 2 (opponent) plays, this strategy is worse than or equal to **C** (i.e., **C** *weakly dominates* **A**)

    → *no rational player would ever play* **A**!
  - **B**: high payoff combinations (**B**,**W**) and (**B**,**Z**)

    → *works only in case players cooperate*!
  - **B** and **C**: highest sum of possible payoffs (i.e., row sums)

    → *best only if the opponent picks her strategy at random*!
  - **C**: looks safe – always gives the highest or second-highest payoff

    → *doesn't take the opponents rationality into account*!
- *game-theoretic approach*: both players rational, aware of the other player's rationality
  - optimal strategy: **D** (we'll be explaining why in the following lectures)

Battle of the Networks: A Constant-Sum Game

- suppose there are just two television networks. Both are battling for shares of viewers (0 – 100%). Higher shares are preferred (= higher advertising revenues).
  - sum of shares = 100%, i.e. for two players

$$Z_1(x_1,x_2) + Z_2(x_1,x_2) = const. \quad \text{for all } (x_1,x_2)$$

- network 1 has an advantage in sitcoms. If it runs a sitcom, it always gets a higher share than if it runs a game show.
- network 2 has an advantage in game shows. If it runs a game show it always gets a higher share than if it runs a sitcom.

Network 2

Network 1

1 \ 2	Sitcom	Game show
Sitcom	55% , 45%	52% , 48%
Game show	50% , 50%	45% , 55%

Zero-Sum Games

- **zero-sum game**: a special case of constant-sum games

$$\textit{sum of payoffs} = Z_1 + Z_1 + \ldots + Z_n = 0.$$

- every *constant-sum* game has a *strategically equivalent* counterpart in *zero-sum* games
  - example: zero-sum version of battle of the networks
  - payoffs expressed as the difference from the 50/50 share

  → differences in outcomes unchanged → *strategic equivalence*

Network 2

	1 \ 2	Sitcom	Game show
Network 1	Sitcom	5% , -5%	2% , -2%
	Game show	0% , 0%	-5% , 5%

Note: entries for 1 and 2 always opposite ($Z_1 = -Z_2$) → *no need to write both!*

Matrix Games

- a special case of zero-sum games:
  - a finite set of agents: $\{1,2\}$
  - strategy spaces (*finite*): $\{X, Y\}$
    - strategy profile: (x,y)
  - payoff functions: $Z_1(x,y)$, $Z_2(x,y)$
  - zero-sum payoffs: $Z_1(x,y) + Z_2(x,y) = 0$

- payoffs written in a matrix, typically denoted by $\boldsymbol{A}$:

$$\boldsymbol{A} = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- a_{ij} = *the payoff of player* 1 for strategy profile (i,j)
  (i.e., player 1 picks ith strategy and player 2 picks jth)

Matrix Games (cont'd)

- *example*: battle of the networks (zero-sum version)
  - a matrix game with

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 0 & -5 \end{bmatrix}$$

  - note: in order to know the strategic nature of the game, nothing else needs to be specified (the payoffs and number of strategies of both players are determined by **A**)

Network 2

Network 1

1 \ 2	Sitcom	Game show
Sitcom	5	2
Game show	0	-5

Nash Equilibrium

- one of the most widely used game-theoretical concepts (not only for matrix games)
- *best-response approach*:
  - determine the "best response" of each player to a particular choice of strategy by the other player (do this for both players)
  - if each player's strategy choice is a best response to the strategy choice of the other player, we're in a *Nash equilibrium* (NE)

"*NE is such a combination of strategies that neither of the players can increase their payoff by choosing a different strategy.*"

"*NE is a solution with the property that whoever of the players chooses some other strategy, he or she will not increase his or her payoff.*"

Nash Equilibrium in Matrix Games

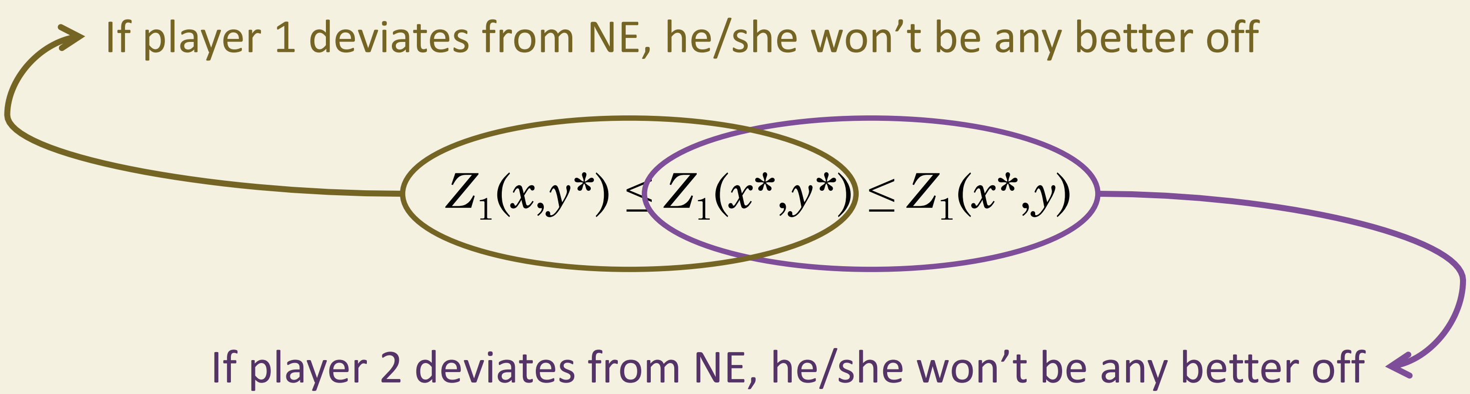
- **mathematical definition**:

  A strategy profile (x^*, y^*) with the property that

$$Z_1(x,y^*) \leq Z_1(x^*,y^*) \leq Z_1(x^*,y)$$

  for all $x \in X$ and $y \in Y$ is a NE.

Inequality from the definition above explained:



Finding NE's in Matrix Games

- a matrix game can have 0, 1 or multiple NE's
- **best-response analysis** (a.k.a. cell-by-cell inspection)
  - network 1's best response:
    - if network 2 runs a sitcom, network 1's best response is to run a sitcom. Circle (*S*,*S*).
    - if network 2 runs a game show, network 1's best response is to run a sitcom. Circle (*S*,*G*).

Network 2

Network 1

1 \ 2	S	G
S	(5)	(2)
G	0	-5

Finding NE's in Matrix Games (cont'd)

- network 2's best response:
  - if network 1 runs a sitcom, network 2's best response is to run a game show. Square (S,G).
  - if network 1 runs a game show, network 2's best response is to run a game show. Square (G,G).
- the NE strategy profile is (S,G). (if network 2 plays G, network 1's best response is S and vice versa)

Network 2

Network 1

1 \ 2	S	G
S	5	2
G	0	-5

Finding NE's in Matrix Games (cont'd)

- from the best-response analysis it follows that a NE is represented by such an element in the payoff matrix that is both…
  - …the *maximum* in its column (player 1's best response)
  - …the *minimum* in its row (player 2's best response)
- such an element is called a **saddle point** of the matrix
- value of a saddle point = $Z_1(x^*,y^*)$ = **value of the game**
- notion of *stability*: neither player has an incentive to deviate from NE

NE's in Matrix Games: Exercise 1

- find a NE in the following matrix game:

Player 2

Player 1

1 \ 2	W	X	Y	Z
A	4	4	3	5
B	42	10	2	-1
C	-12	56	2	12

NE's in Matrix Games: Exercise 2

- find all NE's in the following matrix game

Player 2

Player 1

1 \ 2	W	X	Y	Z
A	2	3	5	2
B	2	4	5	2
C	-2	7	2	0

Multiple Nash Equilibria

QUIZZ:

Consider a matrix game with payoff matrix $\mathbf{A} = (a_{ij})$. Let a_{27} and a_{43} be two NE's.

a) Is it possible that $a_{27} < a_{43}$?

b) Are a_{23} and a_{47} Nash equilibria as well?

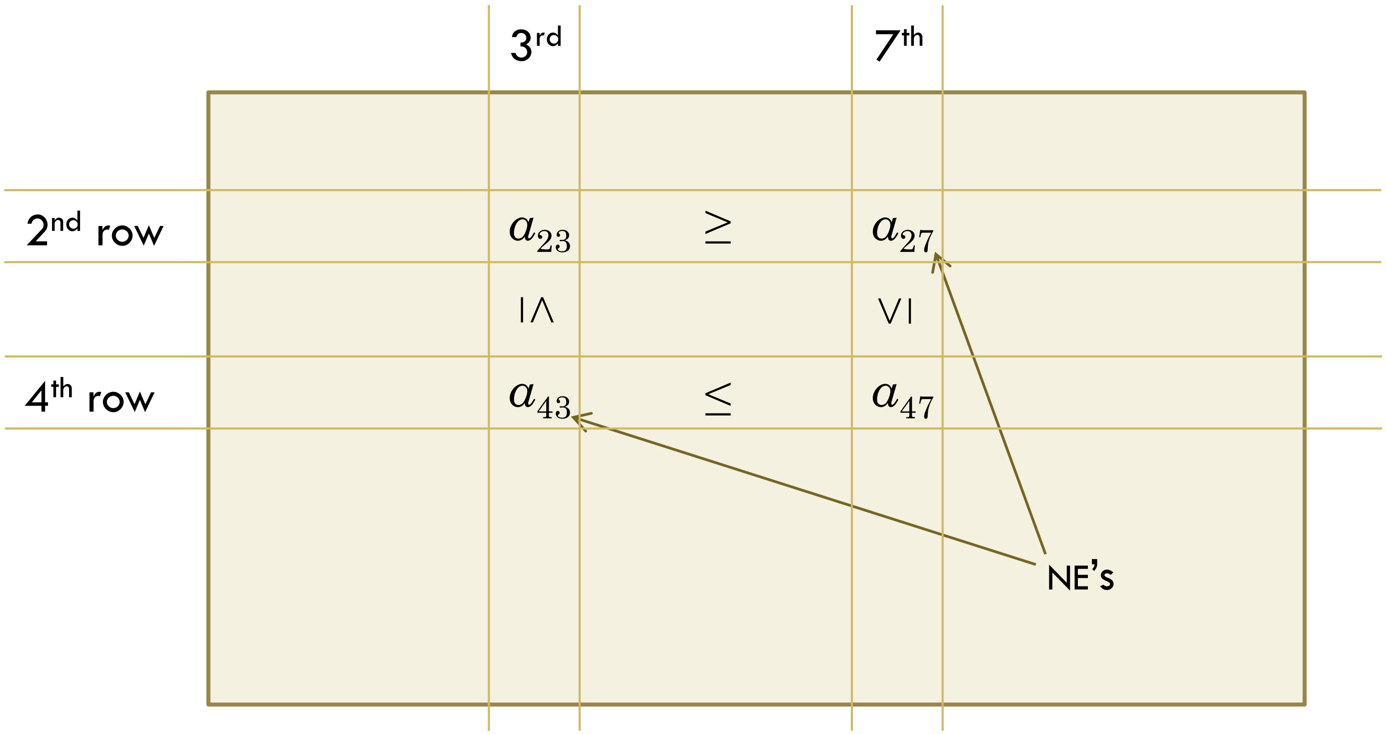
- answers:

  a) no.

  b) yes.

- how to find out: use the basic properties of a saddle point *(see next slide)*

Multiple Nash Equilibria (cont'd)



- *conclusion*: multiple equilibria always have *equal values* and are placed in "rectangular positions"

Dominated strategies

- **definition**:

  Strategy $x_1 \in X$ strictly dominates *strategy* $x_2 \in X$, *if*

  $$Z_1(x_1,y) > Z_1(x_2,y) \text{ for all } y \in Y.$$

  Analogously, $y_1 \in Y$ strictly dominates *strategy* $y_2 \in Y$, *if*

  $$Z_1(x,y_1) < Z_1(x,y_2) \text{ for all } x \in X.$$

  Weak domination *is similar, only it admits* $Z_1(x_1,y) = Z_1(x_2,y)$ *for some* $y \in Y$, *or* $Z_1(x,y_1) = Z_1(x,y_2)$ *for some* $x \in X$.

- example:
  - network 1: G is dominated by S
  - network 2: S is dominated by G

Network 2

Network 1

1 \ 2	S	G
S	5	2
G	0	-5

Dominated strategies (cont'd)

- as a *rational* player would *never* play a *dominated strategy*, matrix games can be simplified by deleting the players' dominated strategies

Network 2

Network 1

1 \ 2	S	G
S	5	2
G	0	-5

- **iterative elimination of dominated strategies**:
  - elimination of dominated rows → columns → rows → columns → …
  - I-know-he-knows-I'm-rational type of thinking

Dominated strategies (cont'd)

- example: extended battle of networks (T = talent show)
  1. network 2: no dominated strategies
  2. network 1: G dominated (by S)
  3. network 2: G dominated (by T)
  4. network 1: T dominated (by S)
  5. network 2: S dominated (by T)

Network 2

Network 1

1 \ 2	S	G	T
S	5	2	1
G	0	-5	-4
T	-2	3	-1

Dominated strategies (cont'd)

- *note: sometimes no strategies to eliminate, but still a single NE, as in the example below*

Network 2

Network 1

1 \ 2	S	G	T
S	5	2	1
G	6	-5	-4
T	-2	3	-1

Dominated strategies: Exercise 3

- eliminate dominated strategies in the matrix game from Exercise 2:

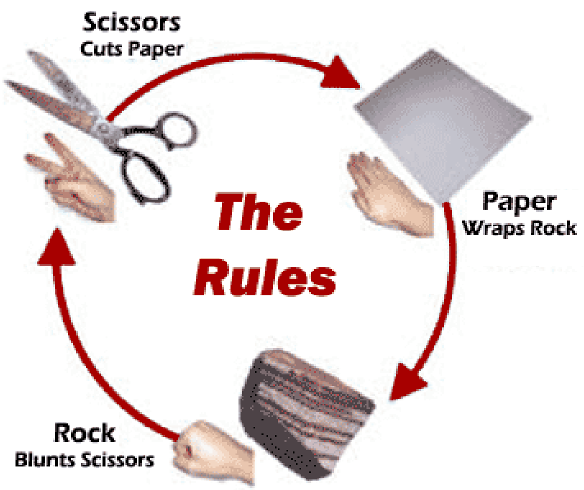
Player 2

Player 1

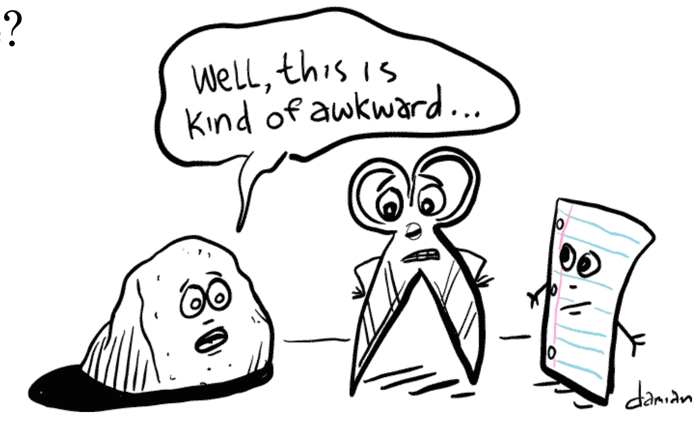
1 \ 2	W	X	Y	Z
A	2	3	5	2
B	2	4	5	2
C	-2	7	2	0

Game 2: Rock, Paper, Scissors

- in this game, each player has 3 strategies: rock (R), paper (P) and scissors (S).
- rules:
  - scissors cut paper
  - paper wraps rock
  - rock crushes scissors
  - winner gets €1 from his/her opponent



a) Can this game be modelled as a matrix game?
b) Which strategy would you choose?
c) Are there any saddle points in the matrix?



Game 2: Rock, Paper, Scissors (cont'd)

Player 2

Player 1

1 \ 2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

- no saddle point in the payoff matrix, but still there's a way to play the game
  - all strategies "equally good" → the best thing for both players is to choose their strategy at random, with equal probabilities
  - even if the other player finds out about the other players' strategy, he/she can't use it against him/her

Game 2: Rock, Paper, Scissors (cont'd)

→ switch from *pure strategies* to *mixed strategies*

- pure strategy: the player decides for a certain strategy
- mixed strategy:
  - the player decides about the probabilities of the alternative strategies
  - when the decisive moment comes, he/she makes a random selection of the strategy with the stated probabilities

- even if a matrix game has no NE in pure strategies (i.e., no saddle point of the payoff matrix), it still has a NE in mixed strategies (*always*)
- optimal mixed strategies for RPC game:

$$\boldsymbol{x}^* = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \quad \boldsymbol{y}^* = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

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