Lecture 11: Games in Extensive Form (cont'd), Bargaining Games

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Games and Decisions

Example 1: Game of Nim (revision)

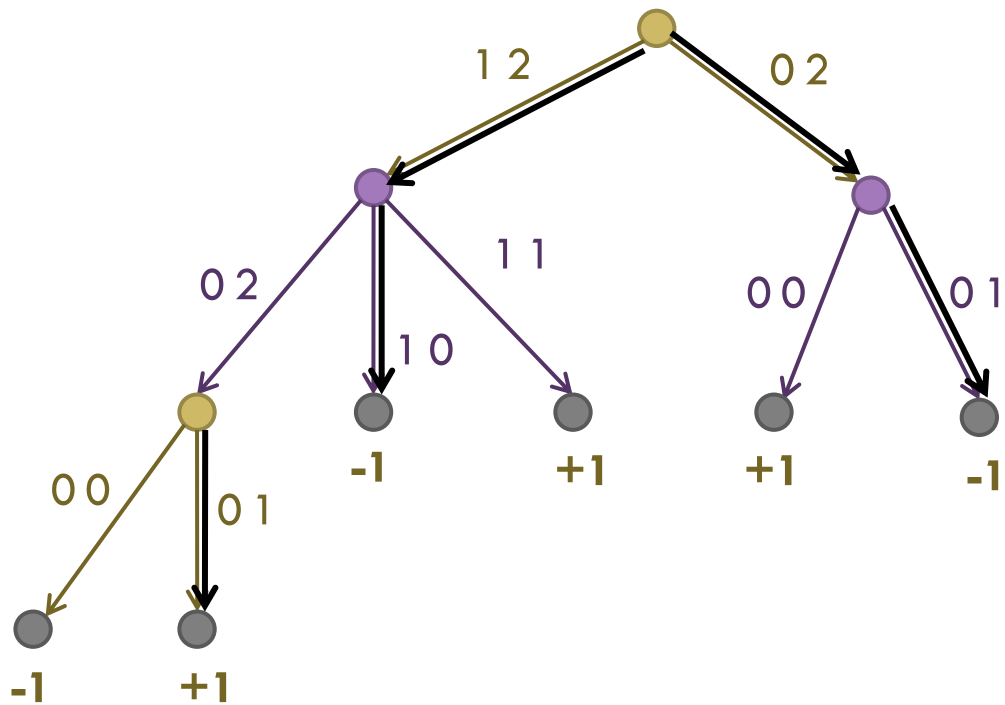
- two players take turns removing objects (matches, tokens) from distinct heaps (piles, rows)
- on each turn, a player must remove an arbitrary number of objects (one or more) from a single heap
- the player to remove the last object loses the game (*zero-sum game*)
- *origins*: centuries ago; mathematical description by Bouton in 1901, the name probably comes from the German word *"nimm"* = *"take!"*
- *notation*: numbers of objects in heaps:

3,4,5 nim game

1,3,5,7 nim game

Example 1: Game of Nim (revision)

- simplified game tree (non-branching nodes omitted):



- player 1 can never win here (unless by fault of player 2)
- simple winning strategy for 2 heaps – *leveling up*: as long as both heaps have at least 2 objects, make them equal size with your move

Nim with More Than 2 Heaps

- for a game with an arbitrary number of heaps, the winning strategy is a bit similar, though slightly more complicated:
  - as long as all heaps have at least 2 objects, make the *nim-sum* of the heaps equal zero in each move (afterwards: odd number of 1s)
  - calculating *nim-sum*:
    1. express the heap counts as sums of powers of 2 $\quad(13 = 8 + 4 + 1)$
    2. cancel out pairs of equal numbers
    3. add up what's left
    - *example*: heaps A,B,C with 3,4,5 objects

			2^2		2^1		2^0					
Heap A:	3	=	0	+	2	+	1	=		2	+	$\cancel{1}$
Heap B:	4	=	4	+	0	+	0	=	$\cancel{4}$			
Heap C:	5	=	4	+	0	+	1	=	$\cancel{4}$		+	$\cancel{1}$
Nim-sum:	2											

Nim with More Than 2 Heaps (cont'd)

- nim-sum = *exclusive-or addition with binary representations*:

Heap A:	011_2	3_{10}
Heap B:	100_2	4_{10}
Heap C:	101_2	5_{10}
Nim-sum:	010_2	2_{10}

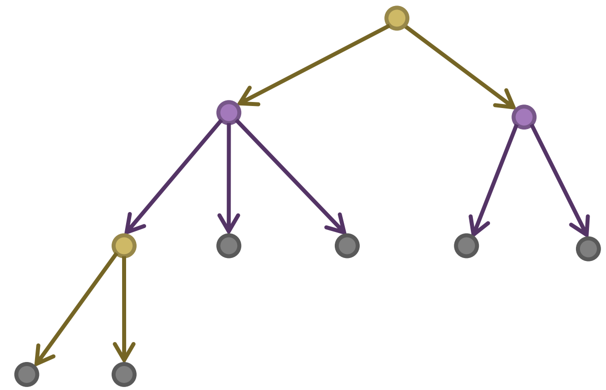
- nim 3,4,5: take 2 from heap *A* (*nim-sum* of 1,4,5 = 0), then make the number of 1s odd (without removing heaps, if possible)
- nim 9,7,5: a winning configuration for player 1

Heap A:	9	=	$8 + 1$
Heap B:	7	=	$\cancel{4} + 2 + \cancel{1}$
Heap C:	5	=	$\cancel{4} + \cancel{1}$
Nim-sum:	11	=	$8 + 2 + 1$

Question: What's the move that makes the nim-sum zero?

Game Complexity

- measure of complexity:
  - number of possible game configurations (state space)
  - *game tree complexity*: number of terminal nodes (= different games)
- *example*: nim 2,2
  - possible configurations: $3 \times 3 =$ **9**
  - game tree complexity: **6**
    (symmetric moves ommited)



- *example 2*: tic-tac-toe
  - state space:
    - simple upper bound: $3^9 = 19{,}683$
      (three states for each of 9 cells)
    - after dropping "illegal" and symmetric/rotated shapes: only 765
  - game tree
    - simple upper bound: $9! = 362{,}880$
      (9 positions for the first move, 8 for the second, and so on.)
  - without illegal/symmetric/rotated: 26,830 possible games

Game Complexity (cont'd)

- as we've seen, games differ with respect to their complexity
  - *very simple*: BoS, Model of Entry, Stackelberg Duopoly
  - *simple*: "small" nim
  - *moderately complex*: tic-tac-toe, "large" nim
  - *complex*: chess

Chess

- every extensive form game with perfect information (such as chess) can be solved using backward induction
  - possible SPNE's: White wins, Draw, White loses
  - empirical evidence suggests either of the first two
- problem with backward induction: game tree way too large, even for computers (today or in future)
- first two moves: $20 \times 20 = 400$ possible games already

Game Complexity (cont'd)

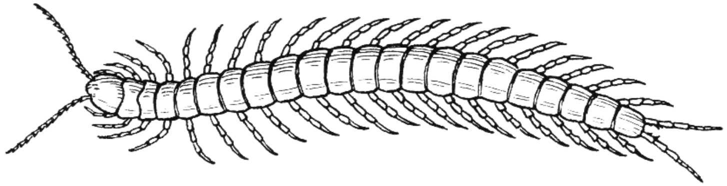
- complexity results:
  - *board positions*: app. 10^{46}
  - *game tree*: app. 10^{123}
  - *compare: number of atoms in observable universe is less than* 10^{81}
- chess software:
  - databases for openings and endings
  - backward induction:
    - several moves ahead only
    - needs a rule for assigning payoffs to non-terminal nodes – an intermediate valuation function (assesses the overall strategic power of a given position)

Large numbers look even larger when written in full:

10^{46} = 10,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000

Exercise 1: Centipede Game

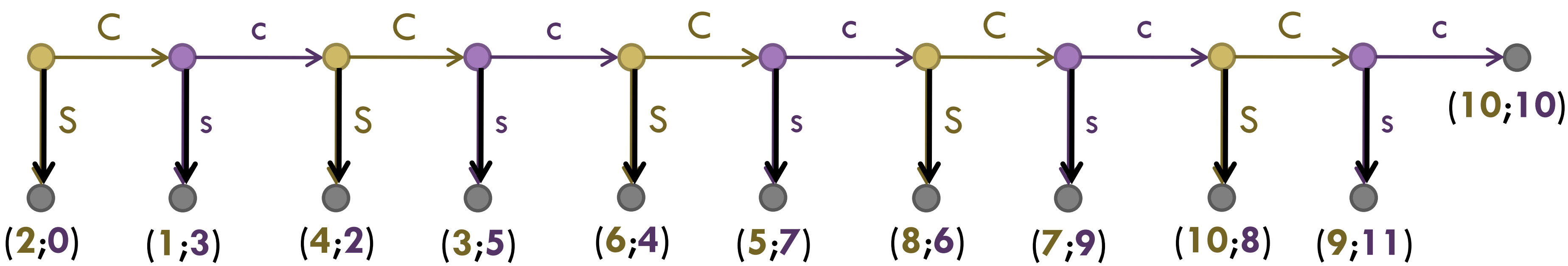
- two players at a table, two heaps of money (initially: \$0 and \$2)
- on his/her move, a player can either:
  - take the larger heap and leave the smaller one for the other player (*stop*, *S*)
  - push the heaps across the table to the other player, which increases both heaps by \$1 (*continue*, *C*)
- this can go on up until 10th round (player 2's 5th move), where instead of increasing the amounts in heaps, the heaps are distributed evenly amongst the players in case of *C*



1. Play the game in pairs.
2. Can you draw the game tree (or part of it, at least)?
3. Try to find the SPNE in the game.

Exercise 1: Centipede Game (cont'd)

- backward induction:



- critique of the SPNE:
  - doesn't reflect the way people behave in complicated games (limited *normativity*)
  - real decision-makers can only go 3-4 nodes "deep"

Bargaining

- an alternative to *fixed price* settings
- the bargaining problem arises in situations where there are possible *gains from trade* (in terms of utility)
  - buyer values the object more than the seller
  - people exchanging goods in a barter trade
- bargaining problems arise when the size of the market is *small and there are no obvious price standards* because the good is *unique, e.g. a house at a particular location, a custom contract to erect a building, etc.*
  - a Seller and a Buyer bargain over the price of a house
  - a Labor Union and Firm bargain over wages & benefits
  - two countries, e.g. the U.S. and Japan bargain over the terms of a trade agreement
  - haggling at informal market settings (common esp. in Asia)
- foundations of bargaining theory: JOHN NASH: The bargaining problem. *Econometrica*, **18**(2), pp. 155–162

Exercise 2: Barter Trade

- Bill and Jack are bartering their goods (without the use of money)
- both maximize their total utility (u for Bill, v for Jack)
- we assume *additive utility functions*: $u(\text{book,whip}) = u(\text{book}) + u(\text{whip})$

		Utility to Bill	Utility to Jack
Bill's goods	book	2	4
	whip	2	2
	ball	2	1
	bat	2	2
	box	4	1
Jack's goods	pen	10	1
	toy	4	1
	knife	6	2
	hat	2	2

1. Play the game in pairs.
2. Is there a way to exchange goods so that both players are better off?

Exercise 2: Barter Trade (cont'd)

- **Nash bargaining solution** (NBS) is a bargaining solution that observes the following principles:
  1. *Pareto optimality*: neither player can be better off without making the other one worse off
  2. *Independence of irrelevant alternatives*:
     - consider two "almost identical" problems (**a** and **b**) that differ only with respect to the set of alternatives, **a** has alternatives in A and **b** in B, such that A is a subset of B; in other words, **a** is a version of **b** with restricted alternatives
       - e.g., imagine **b** is our barter trade example, and in **a** we rule out that Bill gives Jack his box
     - independence of irrelevant alternatives requires that if the bargaining solution to **b** is in A, it is also the bargaining solution to **a**
       - e.g., if in the "best" solution to **b** Bill keeps his box, it must be the "best" solution to **a** as well

Exercise 2: Barter Trade (cont'd)

- Nash proved that NBS is unique (in the sense of resulting utilities) and showed how to find it
  - we'll denote Bill's final collection of items as x and Jack's as y
  - **disagreement value**: if the players fail to agree on any barter exchange, both will get their disagreement value defined as the utility of their original possessions x^0 and y^0
    - Bill: $u(x^0) = u(\text{book,whip,ball,bat,box}) = 2 + 2 + 2 + 2 + 4 = 12$
    - Jack: $u(y^0) = u(\text{pen,toy,knife,hat}) = 1 + 1 + 2 + 2 = 6$
  - **Nash's result**: NBS *is such a combination of x and y that maximizes the product* $[u(x) - u(x^0)]\cdot[v(y) - v(y^0)]$
  - can be found using *MS Excel Solver*

	original owner (=1 if owns)		utility		final owner (=1 if owns)			sum
	Bill	Jack	Bill	Jack	Bill	Jack		sum
book	1	0	2	4	0	1		1
whip	1	0	2	2	0	1		1
ball	1	0	2	1	0	1		1
bat	1	0	2	2	0	1		1
box	1	0	4	1	1	0		1
pen	0	1	10	1	1	0		1
toy	0	1	4	1	1	0		1
knife	0	1	6	2	1	0		1
hat	0	1	2	2	0	1		1

Original:

Bill's utility	12
Jack's utility	6

Final:

Bill's utility	24
Jack's utility	11

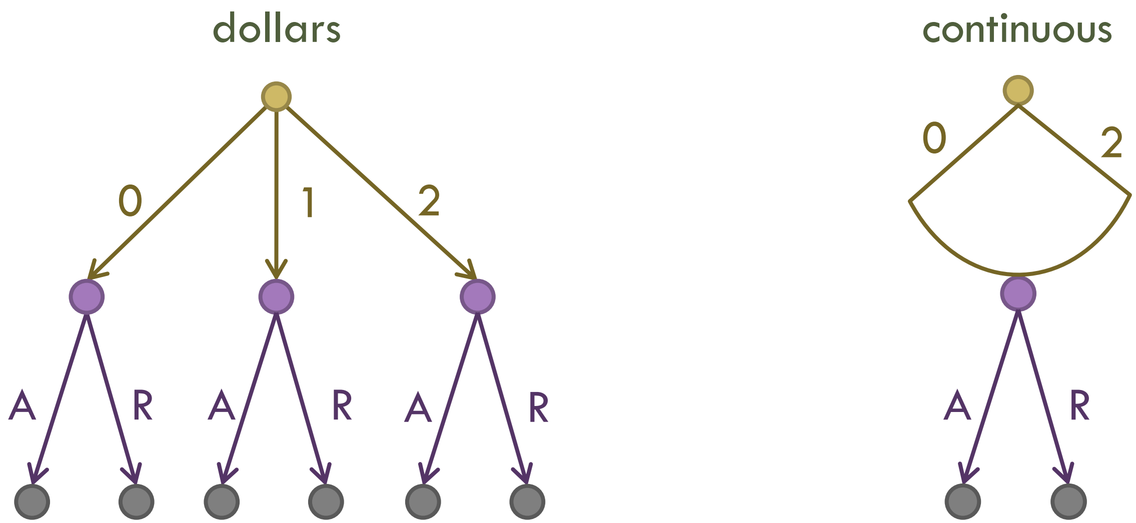
Objective	60

Bill gives Jack: book, whip, ball, bat

Jack gives Bill: pen, toy, knife

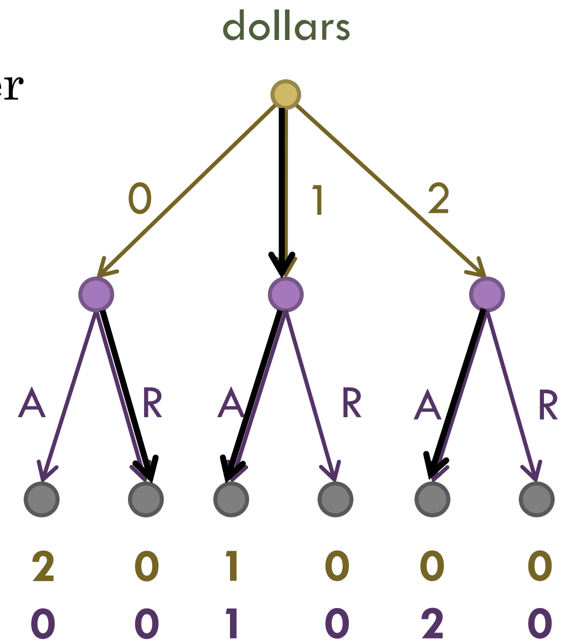
Example 2: Ultimatum Game

- two players interact to decide how to divide a sum of money offered to them (say, \$2)
- player 1 proposes how to divide the sum, player 2 either accepts (A), or rejects (R) – a.k.a. Take-it-or-leave-it
  - if player 2 accepts, player 1's proposal is carried out
  - if player 2 rejects, *neither player receives anything*
- number of possible divisions: dollars, cents or continuous



Example 2: Ultimatum Game (cont'd)

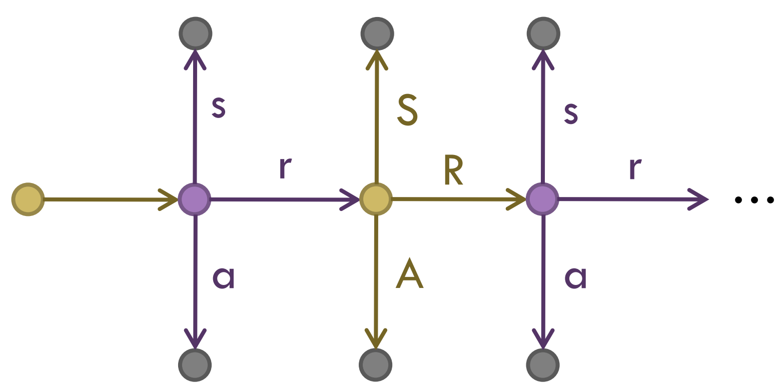
- strategies:
  - player 1: "proposal" number x in [0,10]
  - player 2 (rational strategies): "reject threshold" number y in [0,10]
- equilibria
  - NE: any pair of strategies $x = y$
  - SPNE: $x = y =$ smallest positive number (or $x = y = 0$ if continuous)



Note: empirical studies – shares between 80:20 and 50:50

The Alternating Offers Model

- a sequential move game where players have perfect information, gains of trade (M) are divided among the players
- players take turns making alternating offers, with one offer per round (back-and-forth bargaining)
  - in round 1, player 1 offers a division of M
  - in each consecutive round, the player on the move has three possible actions:
    - *Accept* the other player's offer
    - *Reject* and make a new offer
    - *Stop* the game, thus giving up on bargaining, with both players ending up with their disagreement values (typically 0)



The Alternating Offers Model (cont'd)

When does this end?

- alternating offer bargaining games could continue indefinitely. In reality they do not.
- why not?
  - both sides have agreed to a deadline in advance (or $M = 0$ at a certain date)
    - if deadline = 1st round, we're back to take-it-or-leave-it pricing
  - the gains from trade, M, may *diminish in value over time* (time-limited opportunities), and may fall below disagreement values
    - models with *shrinking factors*: e.g., with each round, M is multiplied by ½.
  - the players are *impatient* (time is money!)
    - future values are discounted
    - practical lesson: act as if you're patient, keep a "poker face", do not respond with counteroffers right away

Class game: Pirates

- a multiplayer version of the Ultimatum game
- rules:
  - There are five rational pirates, A, B, C, D and E. They find 100 gold coins. They must decide how to distribute them.
  - Strict order of seniority: A is superior to B, who is superior to C, who is superior to D, who is superior to E.
  - The pirate world's rules of distribution are thus: first, the most senior pirate should propose a distribution of coins. The pirates, including the proposer, then vote on whether to accept this distribution. If the proposed allocation is approved by a majority or a tie vote, it happens. If not, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to begin the system again.
  - Pirates base their decisions on three criteria (in order of importance
    1. Each pirate wants to survive.
    2. Each pirate tries to maximize the number of gold coins he receives.
    3. Each pirate would prefer to throw another overboard, if all other results would otherwise be equal.

Lecture 11:
Games in Extensive Form (cont'd), Bargaining Games

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Games and Decisions