Black-Scholes model under Arithmetic Brownian Motion

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Abstract

Usually, in the Black-Scholes world, it is assumed that a stock follows a Geometric Brownian motion. The aim of our research is to present Black-Scholes model in a world where the stock is attributed an Arithmetic Brownian motion. Although Arithmetic Brownian motion is simpler due to lack of the geometric terms, as it is shown the option model is eventually analytically less tractable than under Geometric Brownian motion.

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1 Introduction

Black and Scholes in their seminal paper (Black and Scholes, 1973) derived an option pricing model of which one of main assumptions was that underlying stock follows a Geometric Brownian motion (GBM). In this article we are presenting an alternative model that gives one a notion about how would the Black-Scholes look like if Arithmetic Brownian motion (ABM) was used. We are fully aware that in the real-world there is no reason to assume that stock follow ABM but from academic perspective it is an interesting subject to study.

We structure this article as follows. Firstly we equip the reader with simplified main building blocks that determine most of obtained results of the whole article. These are standard theorems and do not differ from those in classical textbooks. Secondly we present a classical model of Black-Scholes, we derive a closed-form formula for call option price. Among many others we apply martingale approach since that is where one can see the beauty of playing with martingales. Of course, we could omit this part related to well-known results but we deem it is a good benchmark for the reader who is interested in the differences in derivation of Black-Scholes under GBM and ABM. Finally, in the third part we derive a call option pricing model on a stock under ABM. We also mention special features of the model and demonstrate closed-form formula for basket options within the ABM framework.

2 Building blocks of option price derivation

2.1 Risk-neutral valuation formula

Although there exist many ways how to derive stock option prices, for the purpose of this study we will use a martingale approach. Martingale approach is based on risk-neutral valuation formula

$$\frac{V_0}{X_0} = \mathbb{E}_X^X \left[ \frac{V_T}{X_T} \right],$$

where option payoff at maturity $V_T = \eta(K, S_T)$ is some deterministic function and $V_0$ is option value as of time $t = 0$. $X$ is a so-called numeraire asset used for relative prices $\frac{V_t}{X_t}$ and also induces a measure $X$. Typically we use a money market account as a numeraire

$$B_t = \exp \left( \int_0^t r_s ds \right) = e^{rt},$$

where we assume $r$ is constant.\(^1\) Such numeraire implies a spot martingale measure $Q$ which will be used through this study. In the world of variety interest-rate derivatives it is often useful to use so-called $T$-forward measure $Q^T$ (see (Jamshidian, 1989) for the original paper), i.e. a measure where a bond $B(\cdot, T)$ is a numeraire. In our case of constant interest rates both approaches coincide.

Since we use the money market account $B$ as the numeraire the risk-neutral valuation formula that we use through this study is

$$\frac{V_0}{B_0} = \mathbb{E}_Q^Q \left[ \frac{V_T}{B_T} \right]. \quad (2.1)$$

Bear also in mind that by definition $B_0 = 1$ and due to constant risk-free rate $B_T = e^{rT}$. The formula (2.1) can thus be rewritten as

$$V_0 = e^{-rT} \mathbb{E}_Q^Q [V_T]. \quad (2.2)$$

Since we derive a call option price we set

$$V_T = \max[S_T - K, 0]. \quad (2.3)$$

\(^1\)in terms of stock option pricing this is an acceptable compromise
2.2 Stock motion $S$ under measure $Q$

Normally, the stock motion $S$ under empirical measure $\mathbb{P}$ has the dynamics

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW^\mathbb{P}_t. \quad (2.4)$$

We will specialize further this Ito’s process so that it models either GBM or ABM but now it is preferable to work with the generic form (2.4).

The question implied by (2.1)\(^2\) (or by (2.2), respectively) is what is the $Q$-dynamics of $S$? There are several useful definitions of spot martingale measure $Q$ in (Björk, 2009) and of these we will suffice with the definition that under $Q$ the price process $S_t$ must be a martingale which means a driftless process.

Set $U_t = B_t^{-1}$ and consider a differential of the process $Z_t = Z(S_t, U_t) = S_t U_t$. Ito’s product rule (see (Joshi, 2003, Eq. (5.34))) yields

$$dZ_t = S_t dU_t + U_t dS_t + dS_t dU_t = -rS_t e^{-rt} dt + e^{-rt} dS_t + \left[ \cdots \right].$$

In order to kill the drift of $Z_t$ one must therefore ensure that $(\mu(S_t, t) - rS_t) = 0$. This is done by a measure change from $\mathbb{P}$ to $Q$ via a Girsanov theorem. Since the change of measure for ABM differs from that for GBM we specialize the form of Girsanov transformation in the sections related to ABM and GBM models.

To this end let

$$L_t = \left( \frac{dQ}{d\mathbb{P}} \mid \mathcal{F}_t \right),$$

denote the Radon-Nikodym derivative. Then we have the following Radon-Nikodym process

$$dL_t = L_t \varphi(t) dW^\mathbb{P}_t,$$

where $\varphi(t)$ is a Girsanov transformation kernel. Girsanov kernel gives us a new $Q$-Wiener process given by

$$dW^Q_t = \varphi(t) dt + dW^Q_t. \quad (2.5)$$

We will use (2.5) quite extensively because it allows to migrate from $\mathbb{P}$ to $Q$.

2.3 Change of numeraire

Consider two geometric processes under measure $Q^0$ ($S^0$ is a numeraire), where $Q^0$ is a martingale measure:

$$dS^0_t = \mu^0 S^0_t dt + \sigma^0 S^0_t dW^0_t,$$

$$dS^1_t = \mu^1 S^1_t dt + \sigma^1 S^1_t dW^0_t. \quad (2.6)$$

Then the Radon-Nikodym derivative takes the form

$$dL_t = L_t (\sigma^1 - \sigma^0) dW^0_t,$$

and via a Girsanov transformation with kernel $(\sigma^1 - \sigma^0)$ defines a new martingale measure $Q^1$ with $S^1$ under $Q^0$ as a numeraire.

\(^2\)we know that $V_T$ is by (2.3) a deterministic function of $S_T$
Proof:
Consider general Ito’s processes under measure $\mathbb{Q}^0$
\[
\begin{align*}
    dS_t^0 &= \mu^0(S_t^0, t)dt + \sigma^0(S_t^0, t)dW_t^0 \\
    dS_t^1 &= \mu^1(S_t^1, t)dt + \sigma^1(S_t^1, t)dW_t^0.
\end{align*}
\tag{2.7}
\]
We know that the Radon-Nikodym derivative that takes us from $\mathbb{Q}^0$ to $\mathbb{Q}^1$ is
\[
dL_t = L_t \varphi(t)dW_t^0,
\]
which is obviously a $\mathbb{Q}^0$–martingale. The change of numeraire is induced when the Radon-Nikodym derivative takes the form
\[
L_t = \frac{S_t^0}{S_t^0} \frac{S_t^1}{S_t^1}.
\tag{2.8}
\]
To get its differential we must use Ito’s lemma. We firstly define an auxiliary process $Z_t = \frac{S_t^1}{S_t^0}$ and so\(^3\)
\[
dZ_t = \frac{S_t^0}{S_t^0} \frac{S_t^1}{S_t^1} dZ_t.
\tag{2.9}
\]
$dZ_t$ now has to be obtained via Ito’s quotient rule (see e.g. (Fries, 2007, Appendix C)) and it yields that
\[
\frac{dZ_t}{Z_t} = \frac{dS_t^1}{S_t^1} - \frac{dS_t^0}{S_t^0} - \frac{dS_t^1}{S_t^1} \frac{dS_t^0}{S_t^0} + \left[ \frac{dS_t^0}{S_t^0} \right]^2,
\]
which is after a substitution for $Z_t$
\[
dZ_t = \frac{S_t^1}{S_t^0} \left( \frac{dS_t^1}{S_t^1} - \frac{dS_t^0}{S_t^0} - \frac{dS_t^1}{S_t^1} \frac{dS_t^0}{S_t^0} + \left[ \frac{dS_t^0}{S_t^0} \right]^2 \right).
\]
By substituting this into (2.9) for $dZ_t$, we get
\[
dL_t = \frac{S_t^0}{S_t^0} \frac{S_t^1}{S_t^1} \left( \frac{dS_t^1}{S_t^1} - \frac{dS_t^0}{S_t^0} - \frac{dS_t^1}{S_t^1} \frac{dS_t^0}{S_t^0} + \left[ \frac{dS_t^0}{S_t^0} \right]^2 \right),
\]
which can be after substitution from (2.8) written as
\[
dL_t = L_t \left( \frac{dS_t^1}{S_t^1} - \frac{dS_t^0}{S_t^0} - \frac{dS_t^1}{S_t^1} \frac{dS_t^0}{S_t^0} + \left[ \frac{dS_t^0}{S_t^0} \right]^2 \right).
\]
Now the problem requires an analysis. We know $L_t$ must be a $\mathbb{Q}^0$–martingale (see for example (Musiela and Rutkowski, 2010) for more details) and so we need to separate and ‘collect’ coefficients of $dt$ in the last expression to drop the drift term. Doing this results into
\[
\begin{align*}
    dL_t &= L_t \left( \frac{\sigma^1(S_t^1, t)dW_t^0}{S_t^1} - \frac{\sigma^0(S_t^0, t)dW_t^0}{S_t^0} \right) \\
    &= L_t \left( \frac{\sigma^1(S_t^1, t)}{S_t^1} - \frac{\sigma^0(S_t^0, t)}{S_t^0} \right) dW_t^0.
\end{align*}
\]
The processes in (2.7), however, are further specified in (2.6) by $\sigma^0(S_t^0, t) = \sigma^0 S_t^0$ and $\sigma^1(S_t^1, t) = \sigma^1 S_t^1$ and plugging these terms into the last equation gives
\[
dL_t = L_t (\sigma^1 - \sigma^0) dW_t^0,
\]
which proves the given result.

\[^3\text{note that } \frac{S_t^0}{S_t^0} \text{ is nothing but a scaling constant}\]
3 Black-Scholes classics: Geometric Brownian Motion

Define

\[ dS_t = \mu S_t dt + \sigma S_t dW_t^P, \quad (3.1) \]

to be Geometric Brownian Motion of a stock \( S \) under the empirical measure \( P \). Our objective is to price a call option \( V_0 \) (we will henceforth assume \( t = 0 \)) on a given non-dividend stock \( S \).

The risk-neutral valuation formula (B as numeraire) given in (2.1) says that we need to obtain dynamics of \( S \) under \( Q \). Following the Section 2.2, with \( \mu(S_t,t) = \mu S_t \) and \( \sigma(S_t,t) = \sigma S_t \) from (3.1) we have the differential of \( Z \) as follows

\[ dZ_t = e^{-rt}([\mu S_t - r S_t] dt + \sigma S_t dW_t^P) \]
\[ = e^{-rt}([\mu - r] S_t dt + \sigma S_t dW_t^P). \quad (3.2) \]

It is clear that if \( \mu = r \) then \( dZ_t \) has zero drift and so \( Z \) would be a martingale under the measure \( Q \) that makes that drift zero. Again following Section 2.2, we set the Girsanov kernel to be

\[ \varphi(t) = -\frac{\mu - r}{\sigma}, \]

and thus by Girsanov change-of-measure we have

\[ dW_t^P = \left( -\frac{\mu - r}{\sigma} \right) dt + dW_t^Q. \quad (3.3) \]

Placing \( dW_t^P \) from (3.3) into (3.2) gives

\[ e^{-rt}([\mu - r] S_t dt + \sigma S_t dW_t^P) = e^{-rt}([\mu - r] S_t dt + \sigma S_t \left( \left( -\frac{\mu - r}{\sigma} \right) dt + dW_t^Q \right)) \]
\[ = e^{-rt} \sigma S_t dW_t^Q, \]

and so indeed \( Z \) is a \( Q \)-martingale. The change of measure also affects the dynamics of the stock in the SDE (3.1) so that

\[ dS_t = \mu S_t dt + \sigma S_t \left( \left( -\frac{\mu - r}{\sigma} \right) dt + dW_t^Q \right), \]

and thus finally after the \( \mu \)-terms cancel out

\[ dS_t = r S_t dt + \sigma S_t dW_t^Q, \quad (3.4) \]

which is a \( Q \)-dynamics of \( S \) crucial to solve the expectation in (2.1).

In order to evaluate (2.1) we could use its form (2.2) so we would be challenged to solve

\[ V_0 = e^{-rT} \int_{-\infty}^{\infty} \max\{S_T - K, 0\} f_{S_T}(S_T) dS_T. \quad (3.5) \]

where \( f_{S_T}(\cdot) \) is a probability density of a random variable \( S_T \). Distribution properties of \( S_T \) are well-known (we omit its derivation here) and we know that\(^4\)

\[ \ln S_T \sim N \left( \ln S_0 + (r - \frac{1}{2} \sigma^2) T, \sigma \sqrt{T} \right), \]

\(^4\)we write \( N(\mu, \sigma) \) for Normal distribution with mean \( \mu \) and standard deviation \( \sigma \)
so
\[ S_T = S_0 \exp \left( \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma W^Q_T \right). \] (3.6)

Now set
\[ Y_T = \left( r - \frac{1}{2} \sigma^2 \right) T + \sigma W^Q_T, \]
so that
\[ Y_T \sim N \left( \left( r - \frac{1}{2} \sigma^2 \right) T, \sigma \sqrt{T} \right), \]
which means we can rewrite \( S_T \) as
\[ S_T = S_0 e^{Y_T}. \] (3.7)

Now we plug (3.7) into (3.5) and it turns out that
\[ V_0 = e^{-rT} \int_{-\infty}^{\max \left[ S_0 e^{Y_T} - K, 0 \right]} f_N(Y_T) dY_T. \]

The max \( \max[\cdot, 0] \) can easily be eliminated when we consider when identify the value of \( Y_T \) when the expression starts to take a positive value. This is valid for \( Y_T > \ln \left( \frac{K}{S_0} \right) \). Now we can rewrite the expression once again, eliminating the \( \max[\cdot, 0] \) and with changed lower bound of integration to have
\[ V_0 = e^{-rT} \int_{\ln \left( \frac{K}{S_0} \right)}^{\infty} (S_0 e^{Y_T} - K) f_N(Y_T) dY_T. \]

A tedious algebra would yield the well-known result. We omit this and use a smarter approach of changing a numeraire.

The trick relies on rewriting (2.1) using indicator functions so that
\[ \frac{V_0}{B_0} = \mathbb{E}^Q \left[ \frac{(S_T - K) 1_{\{S_T \geq K\}}}{B_T} \right]. \]

This equation can be further separated into two terms
\[ \frac{V_0}{B_0} = \mathbb{E}^Q \left[ \frac{S_T 1_{\{S_T \geq K\}}}{B_T} \right] - K \mathbb{E}^Q \left[ \frac{1_{\{S_T \geq K\}}}{B_T} \right]. \]

The expression above can be easily represented in terms of \( J_1 \) and \( J_2 \) such that
\[ \frac{V_0}{B_0} = \frac{J_1}{B_0} - K \frac{J_2}{B_0}, \] (3.8)
where
\[ \frac{J_1}{B_0} = \mathbb{E}^Q \left[ \frac{S_T 1_{\{S_T \geq K\}}}{B_T} \right], \] (3.9)
\[ \frac{J_2}{B_0} = \mathbb{E}^Q \left[ \frac{1_{\{S_T \geq K\}}}{B_T} \right], \] (3.10)
are actually digital options, both expressed in terms of \( Q \)-expectation. \( J_2 \) is easy to solve and we will do this later. Let us now focus on "option" \( J_1 \). The product \( S_T 1_{\{S_T \geq K\}} \) is not trivial so its expectation can
not be directly computed. A trick with change of numeraire can simplify it. Let us now choose the stock \( S \) (under \( Q \)) as the numeraire asset. Now consider our two geometric processes

\[
\begin{align*}
    dB_t &= r B_t dt \\
    dS_t &= r S_t dt + \sigma S_t dW^Q_t.
\end{align*}
\]

Section 2.3 says that the Radon-Nikodym derivative to switch from \( Q \) (induced by numeraire \( B \)) to \( S \) (induced by numeraire \( S \) under \( Q \)) is

\[ dL_t = (\sigma - \sigma_0) L_t dW^Q_t, \]

where we identify Girsanov kernel \( \varphi(t) = \sigma \). Then it follows that

\[ dW^Q_t = \sigma dt + dW^S_t. \]

Substituting the above defined \( dW^Q_t \) into (3.4) gives us SDE for \( S \) under measure with a numeraire \( S \):

\[
    dS_t = (r + \sigma^2) S_t dt + \sigma S_t dW^S_t.
\]

Next, by replacing \( B_0 \) by \( S_0 \) and \( B_T \) by \( S_T \) (we have changed the numeraire from \( B \) to \( S \)) in (3.9) we have

\[
    \frac{J_1}{S_0} = \mathbb{E}^S \left[ \frac{S_T 1_{\{S_T \geq K\}}}{S_T} \right] = \mathbb{E}^S \left[ 1_{\{S_T \geq K\}} \right],
\]

so by the change of numeraire we have effectively eliminated \( S_T \) from the product \( S_T 1_{\{S_T \geq K\}} \). Since

\[
    \mathbb{E}^S \left[ 1_{\{S_T \geq K\}} \right] = S[S_T \geq K],
\]

we can solve this in the following manner. Let us determine the distribution of \( S_T \) under \( S \) by Ito’s lemma for \( dZ(S_t, t) = \ln S_t dS_t - \frac{1}{2} S_t^2 dt \).

This means

\[
    S_T = S_0 \exp \left( \left( r + \frac{1}{2} \sigma^2 \right) T + \sigma W^S_T \right).
\]

Going back to (3.11) let us substitute \( \sqrt{T} X, X \sim N(0, 1) \) for \( W^S_T \) and we have

\[
    \mathbb{E}^S \left[ 1_{\{S_T \geq K\}} \right] = \mathbb{E} \left[ S_0 \exp \left( \left( r + \frac{1}{2} \sigma^2 \right) T + \sigma \sqrt{T} X \right) \geq K \right]
    = \mathbb{E} \left[ X \geq \frac{\ln \left( \frac{K}{S_0} \right) - (r + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right].
\]

Since normal distribution is symmetric, it holds that \( \Pr[X \geq x] \Leftrightarrow \Pr[X < -x] \) and so

\[
    \mathbb{E}^S \left[ 1_{\{S_T \geq K\}} \right] = \mathbb{E} \left[ X \geq \frac{\ln \left( \frac{S_0}{K} \right) + (r + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right]
    = \Phi \left( \frac{\ln \left( \frac{S_0}{K} \right) + (r + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}} \right).
\]
Eventually

\[ J_1 = S_0 N \left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right). \]  

(3.12)

Secondly, in similar fashion we tackle (3.10), which can be rewritten as

\[ J_2 = B_0 e^{-rT} \mathbb{E}^Q \left[ 1_{\{S_T \geq K\}} \right] = e^{-rT} \mathbb{E}^Q \left[ 1_{\{S_T \geq K\}} \right]. \]

We are about to solve

\[ \mathbb{E}^Q \left[ 1_{\{S_T \geq K\}} \right] = \mathbb{Q}[S_T \geq K], \]

where \( S_T \) (under \( \mathbb{Q} \)) has already been defined in (3.6). Analogously to the expectation problem above (now just setting \( W^p_T = X \sqrt{T}, X \sim N(0,1) \)) we solve

\[ \mathbb{E}^Q \left[ 1_{\{S_T \geq K\}} \right] = \mathbb{Q} \left[ S_T \geq K \right] = e^{-rT} \mathbb{Q} \left[ \ln \left( \frac{S_0}{K} \right) \geq \ln \left( \frac{X}{S_0} \right) \right]. \]

Finally

\[ J_2 = e^{-rT} N \left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}} \right). \]  

(3.13)

We can now assemble the Black-Scholes formula. Since from (3.8) if follows that \( V_0 = J_1 - KJ_2 \) we just substitute for \( J_1 \) and \( J_2 \) from (3.12) and (3.13), respectively to have

\[ V_0 = S_0 N(d_1) - Ke^{-rT} N(d_2), \]

where as usual

\[ d_1 = \frac{\ln \left( \frac{S_0}{K} \right) + \left( r - \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}}, d_2 = d_1 - \sigma \sqrt{T}. \]

4 Black-Scholes under Arithmetic Brownian motion

In the previous section we have derived a classical result - call option price when the underlying asset (stock) follows GBM. Objective of the current section is to derive a call option price under adjusted settings, namely we relax assumption of GBM and adopt assumption of ABM followed by the underlying stock \( S \).

Let us start with SDE for ABM (under \( \mathbb{P} \))

\[ dS_t = \mu dt + \sigma dW^\mathbb{P}_t. \]  

(4.1)

Moving on to the risk-neutral valuation formula, we must make discounted process \( S \) a martingale under \( \mathbb{Q} \). In order to fully apply rules justified in Section 2.2 we must identify our SDE (4.1) with the generic Ito one in (2.4). It is clear that in the case of ABM

\[ \mu(S_t, t) = \mu, \]

\[ \sigma(S_t, t) = \sigma, \]
so the differential of $Z_t = Z(B_t^{-1}S_t)$ takes the form

$$dZ_t = e^{-rt} \left( (\mu - rS_t)dt + \sigma dW_t^Q \right).$$

It is obvious that in order to eliminate the drift the Girsanov kernel that makes $Z_t$ a $\mathbb{Q}$-martingale will not be the same as in the case of GBM.

Set the Girsanov kernel to

$$\varphi(t) = -\frac{\mu - rS_t}{\sigma},$$

which via Girsanov theorem implies

$$dW_t^P = \left( -\frac{\mu - rS_t}{\sigma} \right) dt + dW_t^Q. \quad (4.2)$$

Again recall the differential $dZ_t$ with $dW_t^P$ as defined in (4.2) to have

$$dZ_t = e^{-rt} \left( (\mu - rS_t)dt + \sigma \left[\left(-\frac{\mu - rS_t}{\sigma}\right) dt + dW_t^Q\right]\right) = e^{-rt}\sigma dW_t^Q,$$

from which it is clear that $Z_t = \frac{S_t}{B_t}$ is a $\mathbb{Q}$-martingale.

Combining (4.1) and (4.2) gives us a $\mathbb{Q}$-dynamics of $S$,

$$dS_t = rS_t dt + \sigma dW_t^Q. \quad (4.3)$$

We are, however, also interested in the $\mathbb{Q}$-distribution of $S_T$ associated with the SDE (4.3). The problem is that (4.3) is not geometric so the application of Ito’s lemma to $d\ln S_t$ (as it is usually done for GBM) will not yield a result.

Instead of using logarithm\(^\text{5}\), set $Z_t = Z(S_t, t) = S_t e^{-rt}$ and compute the differential $dZ_t$. By virtue of Ito’s lemma\(^\text{6}\) we have

$$dZ_t = \frac{\partial Z_t}{\partial t} dt + \frac{\partial Z_t}{\partial S_t} dS_t + \frac{1}{2} \left( \frac{\partial^2 Z_t}{\partial (S_t)^2} \right) [dS_t]^2 \quad (4.4)$$

The partials are easily computed as

$$\frac{\partial Z_t}{\partial t} = -rS_t e^{-rt}, \quad \frac{\partial Z_t}{\partial S_t} = e^{-rt}, \quad \frac{\partial^2 Z_t}{(\partial S_t)^2} = 0.$$

Substituting for the partials in (4.4) it turns out that

$$dZ_t = -rS_t e^{-rt} dt + e^{-rt} (rS_t dt + \sigma dW_t^Q) = e^{-rt}\sigma dW_t^Q.$$

Integrating the above between 0 and $T$ we have

$$Z_T - Z_0 = S_T e^{-rT} - S_0 = \int_0^T e^{-rt} \sigma dW_t^Q = \sigma \int_0^T e^{-rt} dW_t^Q.$$

\(^\text{5}\)we use basically the same ‘trick’ that is usually used in Vasicek model when solving for distribution of $r_t$. See (Nawalkha et al., 2007, Example 1.3)

\(^\text{6}\)we can indeed do this because the $S$ under $\mathbb{Q}$-ABM (4.3) complies with the definition of Ito’s process (2.4)
and so
\[ S_T = S_0 e^{rT} + e^{rT} \sigma \int_0^T e^{-rt} dW_t^Q. \] (4.5)

In the last expression we stuck because we can not integrate a deterministic function of \( t \) with respect to \( W \). We can overcome this by using (Björk, 2009, Lemma 4.15) by investigating properties of stochastic integral. The lemma says that if there is some deterministic function \( f(t) \) such that
\[ M(T) = \int_0^T f(t) dW_t, \]
then \( M(T) \) is normally distributed with
\[ E[M(T)] = 0 \]
\[ \text{Var}[M(T)] = \int_0^T f^2(t) dt. \]

Using this rule for our integral in (4.5) gives us
\[ S_T = S_0 e^{rT} + e^{rT} \sigma Y, Y \sim N \left( 0, \sqrt{\frac{1 - e^{-2rT}}{2r}} \right), \]
from which is easy to read off that
\[ S_T = N \left( S_0 e^{rT}, e^{rT} \sigma \sqrt{\frac{1 - e^{-2rT}}{2r}} \right). \] (4.6)

For our later calculations it will be easier to work with
\[ v := e^{rT} \sigma \sqrt{\frac{1 - e^{-2rT}}{2r}}, \]
so that (4.6) is simply just
\[ S_T \sim N \left( S_0 e^{rT}, v \right). \]

Having the \( Q \)-distribution of \( S_T \) we can return to the fundamental pricing problem involving \( Q \)-expectation of \( S_T \). When using our indicator function trick and when we express \( B_0 \) and \( B_T \) explicitly in terms of \( r \) we have
\[ V_0 = e^{-rT} E^Q \left[ \max[S_T - K, 0] \right] = e^{-rT} E^Q \left[ 1_{\{S_T \geq K\}} \max[S_T - K, 0] \right] \]
\[ = e^{-rT} E^Q \left[ 1_{\{S_T \geq K\}} S_T \right] - K e^{-rT} E^Q \left[ 1_{\{S_T \geq K\}} \right]. \] (4.7)

This reminds the \( J_1 \) and \( J_2 \) problem solved in the chapter of GBM. Here, we will not make any further change-of-measure since it is not possible. The reason is that change of measure can relate to positive processes only which rules out ABM.

Let us firstly solve the expectation of the second indicator which is
\[ E^Q \left[ 1_{\{S_T \geq K\}} \right] = Q[S_T \geq K] \]
\[ = Q \left[ S_0 e^{rT} + vX \geq K \right] = Q \left[ X \geq \frac{K - S_0 e^{rT}}{v} \right] \]
\[ = Q \left[ X \leq \frac{S_0 e^{rT} - K}{v} \right] = N \left( \frac{S_0 e^{rT} - K}{v} \right). \] (4.8)
In the above we used a dummy variable \( X \sim N(0, 1) \) such that \( vX = \sigma e^{rT}Y \).

Now we have to tackle the trickier first expectation of indicator \( E_Q[1_{\{S_T \geq K\}}S_T] \). In the case of GBM we solved this problem by a change-of-numeraire technique which eliminated \( S_T \) from the \( Q \)-expectation \( E_Q[\cdot] \). This technique can not be applied for ABM model because the numeraire price process is required to positive (see (Schönbucher, 2003, Section 4.12) or (Epps, 2007, Definition 3)).

In plain words, we can interpret it as expected value of \( S_T \) in the interval \([K, \infty)\). Expressing this using an integral yields

\[
E_Q[1_{\{S_T \geq K\}}S_T] = \int_K^\infty S_T \frac{1}{v\sqrt{2\pi}} \exp \left( -\frac{(S_T - S_0e^{rT})^2}{2v^2} \right) dS_T. \tag{4.9}
\]

We make a substitution

\[
h_T = \frac{S_T - S_0e^{rT}}{v},
\]

which implies a new lower bound of the last integral

\[
\frac{K - S_0e^{rT}}{v},
\]

and also note that

\[
dh_T = \frac{1}{v} dS_T.
\]

By virtue of the substitution the integral (4.9) changes to

\[
E_Q[1_{\{S_T \geq K\}}S_T] = \int_{\frac{K - S_0e^{rT}}{v}}^\infty S_T \frac{1}{v\sqrt{2\pi}} \exp \left( -\frac{1}{2} \frac{h_T^2}{v^2} \right) dS_T
\]

\[
= \int_{\frac{K - S_0e^{rT}}{v}}^\infty \left[ vh_T + S_0e^{rT} \right] \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \frac{h_T^2}{v^2} \right) dh_T
\]

\[
= S_0e^{rT} \int_{\frac{K - S_0e^{rT}}{v}}^\infty \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \frac{h_T^2}{v^2} \right) dh_T + v \int_{\frac{K - S_0e^{rT}}{v}}^\infty h_T \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} \frac{h_T^2}{v^2} \right) dh_T. \tag{4.10}
\]

The first integral is actually a probability

\[
\Pr \left[ H_T \geq \frac{K - S_0e^{rT}}{v} \right] = \Pr \left[ H_T < \frac{S_0e^{rT} - K}{v} \right] = N \left( \frac{S_0e^{rT} - K}{v} \right), \tag{4.11}
\]

because from the first integral in (4.10) it is clear that we were integrating over standard normal density. The second integral is just a normal density at the lower bound. Thus

\[
f_N \left( \frac{K - S_0e^{rT}}{v} \right) = f_N \left( \frac{S_0e^{rT} - K}{v} \right),
\]

\[
f_N(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right). \tag{4.12}
\]

Using these two last results ((4.11) and (4.12)) we have

\[
E_Q[1_{\{S_T \geq K\}}S_T] = S_0e^{rT}N \left( \frac{S_0e^{rT} - K}{v} \right) + v f_N \left( \frac{K - S_0e^{rT}}{v} \right). \tag{4.13}
\]
Putting together (4.7) with (4.8) and (4.13) yields the final result
\[ V_0 = S_0 N(d) + e^{-rT} [v f_N(d) - KN(d)], \]
where
\[ d = \frac{S_0 e^{rT} - K}{v}. \]

It can be proved that the PDE has the form
\[ \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} = rV. \tag{4.14} \]
This is same as in the case of GBM model except for the coefficient second-order partial derivative which is \( \frac{1}{2} \sigma^2 \) rather than \( \frac{1}{2} \sigma^2 S^2 \) in the case of GBM. The derivation of the PDE (4.14) is shown in Appendix.

4.1 Special features of ABM model

Contrary GBM model, the proposed option value under ABM suggests that there is no solution for the case when \( r = 0 \). This is because \( r \) is in the numerator of \( v \). Since zero interest-rate is not an uncommon observation in the market we need to seek some solution. Let \( d \) depend explicitly on \( r \) so that we for a moment use the notation \( d = d(r) \). Taking limit we obtain
\[ d^* = \lim_{r \to 0} d(r) = \frac{S_0 - K}{\sigma \sqrt{T}}, \]
and so the solution for the option value is simply
\[ V_0 = N(d^*)(S_0 - K) + v f_N(d^*). \]

Finally, we show how simple analytical solution exists for basket options under ABM compared to the case of GBM. The GBM basket option is a demanding pricing problem mostly for the reason that sum of log-normal distributions is not log-normal anymore. This implies that closed-form solution does not exists and so for GBM one has to use some approximation. For instance (Beisser, 1999) uses conditional expectation approach, (Gentle, 1993) approximates the arithmetic average by a geometric average or (Ju, 1993) that uses expansion of ratio of characteristic functions.

The good property of ABM is that we can sum distributions of individual assets in the basket. Having a basket on \( n \) assets \( S = (S_1, S_2, \ldots, S_n) \) for which corresponding weights in the basket are \( w = (w_1, w_2, \ldots, w_n) \) we have the following distribution of the basket portfolio \( Sw^T \).

The covariance matrix is
\[ \Omega = \begin{bmatrix} e^{2rT} \sigma_1^2 \frac{1 - e^{-2rT}}{2r} & \rho_{1,2} e^{2rT} \sigma_1 \sigma_2 \frac{1 - e^{-2rT}}{2r} & \cdots & \rho_{1,n} e^{2rT} \sigma_1 \sigma_n \frac{1 - e^{-2rT}}{2r} \\ \rho_{2,1} e^{2rT} \sigma_2 \sigma_1 \frac{1 - e^{-2rT}}{2r} & e^{2rT} \sigma_2^2 \frac{1 - e^{-2rT}}{2r} & \cdots & \rho_{2,n} e^{2rT} \sigma_2 \sigma_n \frac{1 - e^{-2rT}}{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1} e^{2rT} \sigma_n \sigma_1 \frac{1 - e^{-2rT}}{2r} & \rho_{n,2} e^{2rT} \sigma_n \sigma_2 \frac{1 - e^{-2rT}}{2r} & \cdots & e^{2rT} \sigma_n^2 \frac{1 - e^{-2rT}}{2r} \end{bmatrix}. \]

Thus the basket standard deviation is
\[ \sigma = \sqrt{w \Omega w^T}. \]

Now one can value the basket option using the closed-form formula from the previous section with \( S = Sw^T \) and \( \sigma \) defined above in terms of the covariance matrix \( \Omega \).
5 Conclusion

In this short article we have departed from the usual assumption of GBM for a stock $S$ in Black-Scholes world. Instead, we investigated a pseudo B-S model in which a non-dividend paying stock follows ABM. Indeed, it has been shown that mathematical properties of ABM are less desirable than that of GBM and so the closed-form formula for call option when the stock is following ABM is slightly more difficult to derive. In particular for the ABM case we were unable to invoke the second change of measure as we did in the case of GBM but we had to analytically tackle an integral representing expectation.

We have also investigated special features of the ABM model and shown how easily a basket option can be priced analytically.
A Derivation of the PDE under ABM

Consider a contingent claim $V$ that is function of both stock price $S$ and time $t$ so that we may write $V := V(S, t)$. Using Ito’s lemma we may express differential of $V$ as

$$dV = \left( r \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} \sigma dW_t^Q$$

Form a "hedging” portfolio $\Pi$ such that we short one option and buy $\frac{\partial V}{\partial S}$ (delta) shares $S$:

$$\Pi = \frac{\partial V}{\partial S} S - V.$$ 

Compute the differential of the portfolio value

$$d\Pi = \frac{\partial V}{\partial S} dS - dV.$$ 

After substitution for $dV$ we have

$$d\Pi = \frac{\partial V}{\partial S} dS - \left( r \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \right) dt - \frac{\partial V}{\partial S} \sigma dW_t^Q$$

$$= \frac{\partial V}{\partial S} r dt + \frac{\partial V}{\partial S} \sigma dW_t^Q - \frac{\partial V}{\partial S} r dt - \frac{\partial V}{\partial S} \sigma dW_t^Q = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

Since there is no source of randomness ($W$) in $d\Pi$ anymore, the $\Pi$ portfolio must earn risk-free rate.\(^7\) Thus

$$d\Pi = r \Pi dt.$$ 

Substituting for $d\Pi$ and $\Pi$ we obtain

$$- \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r \left( \frac{\partial V}{\partial S} S - V \right) dt,$$

which is after rearrangement of the terms

$$\frac{\partial V}{\partial t} + r S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial S^2} = r V.$$

\(^7\)here we assume the risk-free rate is earned geometrically. This does not coincide with our assumption of ABM for the stock $S$.
References


